Chapter 2: Construction of Wavelets and Frame Theory

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Orthogonal Wavelet Vectors in a Hilbert Space

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Abstract. In a Hilbert space, some concepts, such as orthogonal wavelet vector, multiresolution analysis(MRA), scaling vector, unitary-shift operator, are introduced, the existence of scaling vectors and orthogonal wavelet vectors are proved, and the standard forms of them are also given. Our abstract arguments give a short and brief proof of the usual existence result of orthogonal wavelet in $L^2(\mathbb{R})$.

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Keywords. Hilbert space; multiresolution analysis; orthogonal wavelet vector; scaling vector.

Wavelet Analysis has been widely used in applied subjects. Abstract Wavelet Analysis has been studied in papers [1-8]. The main aim of the applied Wavelet Analysis is to find wavelets and the usual multiresolution analysis (MRA) is a best way of constructing orthogonal wavelets [9-12]. In this paper, we want to discuss Abstract Multiresolution Analysis (AMAR), i.e., multiresolution analysis in an abstract Hilbert space, which gives a way of constructing orthogonal wavelet vectors. Throughout this paper, $\mathscr{H}$ denotes any separable complex Hilbert space over the complex field $\mathbb{C}$, $\mathscr{B}(\mathscr{H})$ denotes the C^*-algebra consisting of all bounded linear operators on $\mathscr{H}$. For $E \subset \mathscr{B}(\mathscr{H})$, let E' be the commutant of E in $\mathscr{B}(\mathscr{H})$, that is,

$$E' = \{T \in \mathscr{B}(\mathscr{H}) \,|\, TA = AT, \forall\, A \in E\},$$

$\mathscr{U}(E)$ be the set of all unitary operators in E. The letter $\mathbb{Z}$ denotes the set of all integers, and $\ell^2(\mathbb{Z})$ denotes the Hilbert space consisting of all complex sequences $\{c_n\}_{n=-\infty}^{+\infty}$ such that $\sum\limits_{n=-\infty}^{+\infty} |c_n|^2 < +\infty$, $\langle x, y\rangle$ is the inner product of x and y, $\bigvee F$ stands for the closed linear space spanned by F.

Definition 1. If D and T are unitary operators on $\mathscr{H}$ such that $TD = DT^2$, then we call the pair (D, T) a pair of wavelet-operators on $\mathscr{H}$.

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Example 1. (1) Let D be any unitary operator on $\mathscr{H}$, I be the identity operator on $\mathscr{H}$, then (D, I) is a pair of wavelet-operators on $\mathscr{H}$. Generally, if (D, T) is a pair of wavelet-operators on $\mathscr{H}$, then for each unitary U on $\mathscr{H}$, the pair (UDU^*, UTU^*) is also a pair of wavelet-operators on $\mathscr{H}$.

(2) Suppose $\mathscr{H} = L^2(\mathbb{R})$, define $(Df)(t) = \sqrt{2}\, f(2t), (Tf)(t) = f(t-1)$, then (D, T) is a pair of wavelet-operators on $\mathscr{H}$.

(3) Let

$$\mathscr{H} = L^2(\mathbb{R})^n = \{(f_1, f_2, \ldots, f_n) : f_k \in L^2(\mathbb{R})\}$$

be the n-copies of the Hilbert space $L^2(\mathbb{R})$ and (D, T) be as in (2). Define

$$D_n(f_1, f_2, \ldots, f_n) = (Df_1, Df_2, \ldots, Df_n),\ \ T_n(f_1, f_2, \ldots, f_n) = (Tf_1, Tf_2, \ldots, Tf_n),$$

then D_n, T_n are unitaries on $\mathscr{H}$. For any unitary U on the Hilbert space $\mathscr{H}$, it is easy to check that the pair (UD_nU^*, UT_nU^*) is a pair of wavelet-operators on $\mathscr{H}$.

In the following, we use the notation (D, T) to denote an arbitrary pair of wavelet-operators on $\mathscr{H}$. For any unit vector ψ in $\mathscr{H}$, write

$$\psi_{j,\,k} = D^j T^k \psi, \forall\, (j, k) \in \mathbb{Z}^2.$$

Definition 2. If the vectors $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ consists of an orthonormal basis for $\mathscr{H}$, then we say that ψ is a (D, T)-orthogonal wavelet vector in $\mathscr{H}$.

For example, let S be the bilateral shift on the Hilbert space $\ell^2(\mathbb{Z})$ and $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis for $\mathscr{H} = \ell^2(\mathbb{Z})$. Then the vector e_0 is a (S, I)-wavelet vector in $\mathscr{H}$. Also, let h be the Haar wavelet:

$$h(x) = \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2}; \\ -1, & \text{if } \frac{1}{2} \le x < 1; \\ 0, & \text{otherwise,} \end{cases}$$

and (D, T) be as in (2) of Example 1. Then it is well-known that h is a (D, T)-orthogonal wavelet vector in $\mathscr{H}$.

Definition 3. Suppose that $\mathscr{M} = \{V_j \,|\, j \in \mathbb{Z}\}$ is a sequence of closed subspaces of $\mathscr{H}$ satisfying the following conditions:

(1) $V_n \subset V_{n+1}, \forall\, n \in \mathbb{Z}$,

(2) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = \mathscr{H}$,

(3) $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$,

(4) $TV_0 = V_0$,

(5) $D^n V_0 = V_n, \forall\, n \in \mathbb{Z}$,

then $\mathscr{M}$ is called a (D, T)-MRA for $\mathscr{H}$.

Definition 4. Let $\mathscr{M} = \{V_j \,|\, j \in \mathbb{Z}\}$ be a (D, T)-MRA for $\mathscr{H}$. If a unit vector $\varphi \in \mathscr{H}$ such that $\{T^n\varphi \,|\, n \in \mathbb{Z}\}$ consists of an orthonormal basis for V_0, then we say that φ is a (D, T)-scaling vector for $\mathscr{M}$.

Definition 5. Let A be a unitary operator on a Hilbert space $\mathscr{K}$. If there exists a unitary operator $U : \mathscr{K} \longrightarrow \ell^2(\mathbb{Z})$ such that $A = U^*SU$, where S is the bilateral shift on $\ell^2(\mathbb{Z})$, then A is called a U-shift operator on $\mathscr{K}$, shortly, a U-shift.

Proposition 1. *Let $\mathscr{M} = \{V_j \,|\, j \in \mathbb{Z}\}$ be a (D,T)-MRA for $\mathscr{H}$, $T|_{V_0}$ be a U-shift on V_0. Then $\varphi := U^*e_0$ is a (D,T)-scaling vector for $\mathscr{M}$, where $e_0 = \{\delta_{j,0}\}_{j\in\mathbb{Z}}$.*

Theorem 1. *If $\mathscr{M} = \{V_j \,|\, j \in \mathbb{Z}\}$ is a (D,T)-MRA for $\mathscr{H}$, $T|_{V_0}$ is a U-shift on V_0, φ is a unit vector, then φ is a (D,T)-scaling vector for $\mathscr{M}$ if and only if $\varphi \in U^*(\mathscr{U}(\{S\}')e_0)$, where $\mathscr{U}(\{S\}')e_0 = \{Ae_0 \,|\, A \in \mathscr{U}(\{S\}')\}$.*

Proof. Suppose $V \in \mathscr{U}(\{S\}')$, then $VS = SV$ and so $S = VSV^*$. Thus,

$$T|_{V_0} = U^*SU = U^*VSV^*U = (V^*U)^*S(V^*U),$$

which yields that $T|_{V_0}$ is a V^*U-shift. By Proposition 1, U^*Ve_0 is a (D,T)-scaling vector for $\mathscr{M}$.

Conversely, suppose that φ is a (D,T)-scaling vector for $\mathscr{M}$ and $\varphi_1 = U^*e_0$. From Proposition 1 we see that φ_1 is a (D,T)-scaling vector for $\mathscr{M}$. Since φ is also a (D,T)-scaling vector for $\mathscr{M}$, then $\{T^n\varphi_1 \,|\, n \in \mathbb{Z}\}$ and $\{T^n\varphi \,|\, n \in \mathbb{Z}\}$ all are orthonormal bases for Hilbert space V_0. Define $R(T^n\varphi_1) = T^n\varphi$, $\forall n \in \mathbb{Z}$, then we can get a unitary operator R on V_0. Clearly, $RT|_{V_0} = T|_{V_0}R$. Now, we put $V = URU^*$, then V is a unitary operator on $\ell^2(\mathbb{Z})$ and

$$VS = URU^*S = URT|_{V_0}U^* = UT|_{V_0}RU^* = SURU^* = SV,$$

hence $V \in \mathscr{U}(\{S\}')$. In addition, since $U^*Ve_0 = RU^*e_0 = R\varphi_1 = \varphi$, we have $\varphi \in U^*(\mathscr{U}(\{S\}')e_0)$. □

Corollary 1. *If $\mathscr{M} = \{V_j \,|\, j \in \mathbb{Z}\}$ is a (D,T)-MRA for $\mathscr{H}$, $T|_{V_0}$ is a U-shift on V_0, φ is a unit vector, then φ is a (D,T)-scaling vector for $\mathscr{M}$ if and only if $\varphi \in \mathscr{U}(\{T|_{V_0}\}')U^*e_0$.*

Proof. This follows immediately from $\mathscr{U}(\{T|_{V_0}\}')U^* = U^*\mathscr{U}(\{S\}')$. □

Lemma 1. *There exists a unitary operator C_1 on $\ell^2(\mathbb{Z})$ such that*

$$C_1^2 = -I, \ C_1J = JC_1, \ SC_1 = -C_1S^{-1}$$

and

$$|\langle C_1e_k, f\rangle| = |\langle e_{-k-1}, f\rangle| (\forall k \in \mathbb{Z}, \forall f \in \mathscr{H}),$$

where $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the conjugate linear operator on $\ell^2(\mathbb{Z})$ defined by $J\{c_n\} = \{\overline{c_n}\}$.

Proof. Let $\{e_n \,|\, n \in \mathbb{Z}\}$ be the canonical orthonormal basis for $\ell^2(\mathbb{Z})$. It is easy to check that the map

$$C_1 : \sum_{n\in\mathbb{Z}} \lambda_n e_n \mapsto \sum_{n\in\mathbb{Z}} (-1)^{n+1} \ \lambda_{-n-1}\, e_n$$

is a unitary operator on $\ell^2(\mathbb{Z})$ satisfying the desired conditions. Especially, $C_1e_k = (-1)^k e_{-k-1}$. □

Remark 1. For any unitary C_1 on $\ell^2(\mathbb{Z})$ satisfying the four conditions in Lemma 1, let $C = JC_1$, then it can be proved that $C : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bounded conjugate linear operator and has the following properties:

(1) $C^2 = -I$;
(2) $\langle Cx, Cy\rangle = \langle y, x\rangle, \forall x, y \in l^2(\mathbb{Z})$;
(3) $Cx \perp x, \forall x \in \ell^2(\mathbb{Z})$;
(4) $S^n C = (-1)^n CS^{-n}, \forall n \in \mathbb{Z}$;
(5) $|\langle Ce_k, f\rangle| = |\langle e_{-k-1}, f\rangle| (\forall k \in \mathbb{Z}, \forall f \in \mathscr{H})$.

Lemma 2. $\{m - 2k \,|\, k \in \mathbb{Z}\} \cup \{2k - m - 1 \,|\, k \in \mathbb{Z}\} = \mathbb{Z}, \forall m \in \mathbb{Z}$.

Lemma 3. *If $\mathscr{M} = \{V_j \,|\, j \in \mathbb{Z}\}$ is a (D,T)-MRA for $\mathscr{H}$, $T|_{V_0}$ is a U-shift, φ is a (D,T)-scaling vector for $\mathscr{M}$, $f = UD^{-1}\varphi$, and C is any bounded conjugate linear operator satisfying the first four conditions in Remark 1, then*

(1) $\{S^{2l} f \,|\, l \in \mathbb{Z}\}$ is an orthonormal basis for UV_{-1}.
(2) $\mathscr{B} := \{S^{2l} f \,|\, l \in \mathbb{Z}\} \cup \{S^{2l} Cf \,|\, l \in \mathbb{Z}\}$ is an orthonormal subset of $\ell^2(\mathbb{Z})$.

Proof. Since

$$S^{2l} f = S^{2l} UD^{-1}\varphi = UT^{2l} D^{-1}\varphi = UD^{-1} T^l \varphi$$

and $\{T^l \varphi \,|\, l \in \mathbb{Z}\}$ is an orthonormal basis for V_0, then

$$\{S^{2l} f \,|\, l \in \mathbb{Z}\} = UD^{-1} \{T^l \varphi \,|\, l \in \mathbb{Z}\}$$

is an orthonormal basis for $UD^{-1}V_0 = UV_{-1}$. Hence (1) holds. Since φ is a unit vector and S and C all are isometry operators, then each element in $\mathscr{B}$ is a unit vector. Further, it follows from Lemma 1 that

$$S^{2k} Cf = CS^{-2k} f \; (k \in \mathbb{Z}),$$

and thus when $k \neq l$,

$$\langle S^{2k} Cf, S^{2l} Cf\rangle = \langle CS^{-2k} f, CS^{-2l} f\rangle = \langle S^{-2l} f, S^{-2k} f\rangle = 0.$$

Therefore $\{S^{2l} Cf \,|\, l \in \mathbb{Z}\}$ is an orthonormal subset. Moreover, since

$$S^{2k} Cf = S^{k+l} S^{k-l} Cf = (-1)^{k-l} S^{k+l} CS^{l-k} f$$

and by Remark 1, $S^{l-k} f \perp CS^{l-k} f = (-1)^{k-l} S^{k-l} Cf$, we get $S^{l-k} f \perp S^{k-l} Cf$. Since S^{l+k} is unitary, so

$$S^{2l} f = S^{k+l} S^{l-k} f \perp S^{k+l} S^{k-l} Cf = S^{2k} Cf.$$

It follows from part (1) that $\mathscr{B}$ is an orthonormal subset in $\ell^2(\mathbb{Z})$. □

Lemma 4. *If $E, F \subset \mathscr{H}$ and $E \perp F$, then $\bigvee(E \cup F) = \bigvee E \oplus \bigvee F$.*

Proposition 2. *Let $\mathscr{M} = \{V_j \,|\, j \in \mathbb{Z}\}$ be a sequence of closed linear subspaces of $\mathscr{H}$ satisfying the conditions (1)∼(3) in Definition 3, and $W_n = V_{n+1} \cap V_n^{\perp} = V_{n+1} \ominus V_n$. Then*

$$\mathscr{H} = \underset{n\in\mathbb{Z}}{\oplus} W_n = \left\{ \sum_{n\in\mathbb{Z}} x_n \,|\, x_n \in W_n \text{ and } \sum_{n\in\mathbb{Z}} x_n \text{ is convergent} \right\}.$$

Now, we can give a method of constructing (D,T)-orthogonal wavelet vectors from a given (D,T)-MRA.

Theorem 2. *If $\mathscr{M}=\{V_j\,|\,j\in\mathbb{Z}\}$ is a (D,T)-MRA for $\mathscr{H}$, $T|_{V_0}$ is a U-shift on V_0, φ is a (D,T)-scaling vector for $\mathscr{M}$, then for each integer k, $\psi_k:=DU^*S^{2k}CUD^*\varphi$ is a (D,T)-orthogonal wavelet vector in $\mathscr{H}$, where C is any bounded conjugate linear operator having the properties (1)-(5) in Remark 1.*

Proof. By Proposition 2, $\mathscr{H}=\bigoplus_{n\in\mathbb{Z}} W_j$. It is easy to see that

$$W_j=V_{j+1}\cap V_j^{\perp}=(D^jV_1)\cap(D^jV_0)^{\perp}=D^j(V_1\cap V_0^{\perp})=D^jW_0,$$

hence $\mathscr{H}=\bigoplus_{j\in\mathbb{Z}} D^jW_0$. So it suffices to show that $\{T^l\psi_k\,|\,l\in\mathbb{Z}\}$ is an orthonormal basis for W_0. Put $f=UD^{-1}\varphi$, then

$$T^l\psi_k=T^lDU^*S^{2k}CUD^{-1}\varphi=DT^{2l}U^*S^{2k}CUD^{-1}\varphi=DU^*S^{2l+2k}Cf.$$

Thus, to prove that $\{T^l\psi\,|\,l\in\mathbb{Z}\}$ is an orthonormal basis for W_0, we only need to show that $\{U^*S^{2l}Cf\,|\,l\in\mathbb{Z}\}$ is an orthonormal basis for W_{-1} (because $DW_{-1}=W_0$). Further, because $U^*(UV_0\ominus UV_{-1})=V_0\ominus V_{-1}=W_{-1}$, it suffices to show that $\{S^{2l}Cf\,|\,l\in\mathbb{Z}\}$ is an orthonormal basis for $UV_0\ominus UV_{-1}$. By Lemma 3, the set

$$\mathscr{B}:=\{S^{2l}f\,|\,l\in\mathbb{Z}\}\cup\{S^{2l}Cf\,|\,l\in\mathbb{Z}\}$$

is an orthonormal set in $\ell^2(\mathbb{Z})=UV_0$. Combining with Lemma 4, we see that

$$\bigvee\mathscr{B}=\bigvee\{S^{2l}f\,|\,l\in\mathbb{Z}\}\oplus\bigvee\{S^{2l}Cf\,|\,l\in\mathbb{Z}\}=UV_{-1}\oplus\bigvee\{S^{2l}Cf\,|\,l\in\mathbb{Z}\}.$$

Suppose that P is the orthogonal projection of $\ell^2(\mathbb{Z})$ onto $\bigvee\mathscr{B}$, and $f=\sum_{n\in\mathbb{Z}}\lambda_ne_n$, then

$$|\langle e_m,S^{2l}f\rangle|=\left|\left\langle e_m,\sum_{n\in\mathbb{Z}}\lambda_{n-2l}\,e_n\right\rangle\right|=|\lambda_{m-2l}|,$$

$$|\langle e_m,S^{2l}Cf\rangle|=|\langle Ce_m,S^{-2l}f\rangle|=|\langle e_{-m-1},S^{-2l}f\rangle|=|\lambda_{2l-m-1}|.$$

It follows from Lemma 2 that

$$\begin{aligned}\|Pe_m\|^2&=\sum_{l\in\mathbb{Z}}|\langle e_m,S^{2l}f\rangle|^2+\sum_{l\in\mathbb{Z}}|\langle e_m,S^{2l}Cf\rangle|^2\\&=\sum_{l\in\mathbb{Z}}|\lambda_{m-2l}|^2+\sum_{l\in\mathbb{Z}}|\lambda_{2l-m-1}|^2\\&=\|f\|^2\\&=1.\end{aligned}$$

Hence $\|e_m-Pe_m\|^2=\|e_m\|^2-\|Pe_m\|^2=0$. This implies that $Pe_m=e_m,\forall m\in\mathbb{Z}$. Thus, $\ell^2(\mathbb{Z})=\bigvee\mathscr{B}$. Therefore

$$\bigvee\{S^{2l}Cf\,|\,l\in\mathbb{Z}\}=(UV_0)\cap(UV_{-1})^{\perp}=UV_0\ominus UV_{-1}.$$

This shows that $\left\{S^{2l}Cf \mid l \in \mathbb{Z}\right\}$ is an orthonormal basis for $UV_0 \ominus UV_{-1}$. Hence, from preceding arguments we can conclude that ψ_k is a (D,T)-orthogonal wavelet vector in $\mathscr{H}$. □

Next theorem gives a general form of (D,T)-orthogonal wavelet vectors.

Theorem 3. *Let ψ_0 be a (D,T)-orthogonal wavelet vector in $\mathscr{H}$, ψ be a unit vector in $\mathscr{H}$, then ψ is a (D,T)-orthogonal wavelet vector in $\mathscr{H}$ if and only if there exists $V \in \mathscr{U}(C_{\psi_0}\{D,T\})$ such that $\psi = V\psi_0$, where $C_{\psi_0}\{D,T\}$ stands for the set*

$$\{A \in \mathscr{B}(\mathscr{H}) \mid AB\psi_0 = BA\psi_0, \forall\, B \in \{D,T\}\}.$$

Proof. Suppose $V \in \mathscr{U}(C_{\psi_0}\{D,T\}), \psi = V\psi_0$, then

$$D\psi = DV\psi_0 = VD\psi_0,\ T\psi = TV\psi_0 = VT\psi_0.$$

Hence

$$\psi_{j,k} = D^jT^k\psi = VD^jT^k\psi_0 \text{ for } j,k \in \mathbb{Z}.$$

Since $\left\{D^jT^k\psi_0 \mid j,k \in \mathbb{Z}\right\}$ is an orthonormal basis for $\mathscr{H}$ and V is unitary, so $\{\psi_{j,k} \mid j,k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathscr{H}$ and then ψ is a (D,T)-orthogonal wavelet vector in $\mathscr{H}$.

On the other hand, suppose that ψ is a (D,T)-orthogonal wavelet vector in $\mathscr{H}$, then $\left\{D^jT^k\psi_0 \mid j,k \in \mathbb{Z}\right\}$ and $\left\{D^jT^k\psi \mid j,k \in \mathbb{Z}\right\}$ all are orthonormal bases for $\mathscr{H}$. Define

$$V(D^jT^k\psi_0) = D^jT^k\psi\,(j,k \in \mathbb{Z}),$$

then we obtain a unitary operator V on $\mathscr{H}$ with $V\psi_0 = \psi$. Further, since

$$DV\psi_0 = D\psi = VD\psi_0,$$

$$TV\psi_0 = T\psi = VT\psi_0,$$

we see that $V \in \mathscr{U}(C_{\psi_0}\{D,T\})$ which satisfies $\psi = V\psi_0$. □

Example 2. Let $\mathscr{H}$ and (D,T) be as in part (2) of Example 1, define $\varphi_0 = \chi_{[0,1)}$, and

$$V_0 = \bigvee\{T^n\varphi_0 \mid n \in \mathbb{Z}\} = \bigvee\left\{\chi_{[n,n+1)} \mid n \in \mathbb{Z}\right\}, V_n = D^nV_0,$$

Then $\mathscr{M} = \{V_n \mid n \in \mathbb{Z}\}$ is a (D,T)-MRA for $\mathscr{H}$([11, Example 6.4.3]). Define $U : V_0 \longrightarrow \ell^2(\mathbb{Z})$ by $U(T^n\varphi_0) = e_n$, then U is a unitary operator with $UT|_{V_0} = SU$. So $T|_{V_0}$ is a U-shift on V_0. It is easy to check that φ_0 is a (D,T)-scaling vector for $\mathscr{M}$ and $\varphi_0 = U^*e_0$. By Theorem 2, $\psi = DU^*CUD^*\varphi_0 = \chi_{[-1/2,0)} - \chi_{[-1,-1/2)}$ is a (D,T)-orthogonal wavelet vector in $\mathscr{H}$. It is clear that $-T\psi$ is also a (D,T)-orthogonal wavelet vector in $\mathscr{H}$. Clearly, $-T\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, which is just the Haar wavelet.

By using our abstract result(Theorem 2), we can easily deduce the following standard result in traditional wavelet analysis, without using Fourier transformation.

Theorem 4. *Let $\mathscr{M} = \{V_n \mid n \in \mathbb{Z}\}$ be an MRA for the Hilbert space $L^2(\mathbb{R})$ with a scaling function φ that makes $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0. Then for every integer k, the function*

$$\psi_k = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \langle \varphi_{1,-n-1}, \varphi \rangle \varphi_{1,n+2k}$$

is an orthogonal wavelet in $L^2(\mathbb{R})$, where $\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k)$.

Proof. Let $\mathscr{H} = L^2(\mathbb{R})$ and (D,T) be as in the part (2) of Example 1, then $\mathscr{M} = \{V_n \mid n \in \mathbb{Z}\}$ is a (D,T)-MRA for the Hilbert space $\mathscr{H}$ with a (D,T)-scaling vector φ. Define $U : V_0 \longrightarrow \ell^2(\mathbb{Z})$ by $U(T^n\varphi) = e_n$, then U is a unitary operator with $UT|_{V_0} = SU$. So $T|_{V_0}$ is a U-shift on V_0.

Let C_1 be the unitary operator given by the proof of Lemma 1, and $C = JC_1$, then

$$C\left(\sum_{n\in\mathbb{Z}} \lambda_n e_n\right) = \sum_{n\in\mathbb{Z}} (-1)^{n+1}\, \overline{\lambda_{-n-1}} e_n,$$

for all $\{\lambda_n\}_{n\in\mathbb{Z}}$. From Theorem 2, we see that $\psi_k = DU^*S^{2k}CUD^*\varphi$ is a (D,T)-orthogonal wavelet vector in $\mathscr{H}$ for every integer k. Since $D^*\varphi \in V_{-1} \subset V_0$ and $\{T^n\varphi\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_0, we see that

$$D^*\varphi = \sum_{n\in\mathbb{Z}} \langle D^*\varphi, T^n\varphi \rangle T^n\varphi.$$

Thus, an easy computation shows that

$$\psi_k = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \langle DT^{-1-n}\varphi, \varphi \rangle DT^{n+2k}\varphi = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \langle \varphi_{1,-n-1}, \varphi \rangle \varphi_{1,n+2k}.$$

This completes the proof. □

Remark 2. Under the assumptions of Theorem 4, since $T^kD = DT^{2k}$, we see that $\psi_k = T^k\psi_0$ for every integer k. Especially, the functions

$$\psi_0 = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \langle DT^{-1-n}\varphi, \varphi \rangle DT^n\varphi \text{ (See [12], P. 135)}$$

and

$$\psi_1 = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \langle DT^{-1-n}\varphi, \varphi \rangle DT^{n+2}\varphi = \sum_{n\in\mathbb{Z}} (-1)^{n-1} \langle DT^{1-n}\varphi, \varphi \rangle DT^n\varphi$$

are orthonormal wavelets in $L^2(\mathbb{R})$.

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Operator Frames for $B(\mathcal{H})$

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Abstract. Operator frames for the space $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ are introduced and discussed. By introducing the concept of operator response of vectors in a Hilbert space, we establish a relationship between operator frames for $B(\mathcal{H})$ and usual frames for Hilbert space $\mathcal{H}$ and show that operator frames preserve so many properties of usual frames that we can say that the concept of operator frames is a generalization of frames for Hilbert spaces. In fact, a frame for a Hilbert space or a frame of subspaces for a Hilbert space may be considered as a special case of operator frames in certain sense.

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Keywords. Operator frame; operator Riesz basis; Hilbert space.

1. Introduction

In 1946, Gabor [12] discussed a decomposition of signals in terms of elementary signals. In 1952, Duffin and Schaeffer [11] generalized the Gabor's fundamental ideal and introduced the notion of frame in nonharmonic Fourier analysis. The work of Duffin and Schaeffer was not continued until 1986 when Daubechies, Grossman and Meyer [10] applied the theory of frame to wavelet and Gabor transform. After their work, the theory of frame began to be studied widely and deeply. Today, the theory of frame has been applied to signal processing, image processing, data compressing, and sampling theory and so on.

In this paper, let Λ be the set of all integral numbers or the set of all natural numbers, $\mathcal{F}(\Lambda)$ be the set of all nonempty finite subsets of Λ and $B(\mathcal{H}, \mathcal{K})$ be the set of all bounded linear operators from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. In the case where $\mathcal{K} = \mathcal{H}$, we write $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$.

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A frame for a Hilbert space $\mathcal{H}$ is a sequence $\{f_i\}_{i\in\Lambda}$ of vectors in $\mathcal{H}$ satisfying the condition that there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in\Lambda} |\langle f_i, x\rangle|^2 \leq B\|x\|^2, \forall x \in \mathcal{H}.$$

Theory of frames for Hilbert spaces have been developed deeply. Many people have done momentous works in this field such as D. Han, D. Lason, R. Young, P. G. Casazza, O. Christensen and H. X. Cao(see [2,6,9,14,16]).

In 2004, P. G. Casazza introduced in [3] the notion of frames of subspaces. Let $\{v_i\}_{i\in\Lambda}$ be a family of weights, i.e. $v_i > 0$, $\forall i \in \Lambda$. A family of subspaces $\{W_i\}_{i\in\Lambda}$ of a Hilbert space $\mathcal{H}$ is a frame of subspaces with respect to $\{v_i\}_{i\in\Lambda}$ for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in\Lambda} v_i^2 \|\pi_{W_i} x\|^2 \leq B\|x\|^2, \forall x \in \mathcal{H}, \tag{1.1}$$

where π_{W_i} is the projection from $\mathcal{H}$ onto $W_i, \forall i \in \Lambda$.

In 1990s, Grochenig, Aldroubi, Sung and Tang began to study the theory of frame in Banach spaces. They introduced two kinds of notions of frames in a Banach space: Banach frames and $p-$frames$(1 < p < \infty)$. A sequence $\{f_i^*\}_{i\in\Lambda}$ in the dual space X^* of a Banach space X is a $p-$frame for X if there exist constants $A, B > 0$ such that

$$A\|x\| \leq \|\{\langle x, f_i^*\rangle\}_{i\in\Lambda}\|_{l^p(\Lambda)} \leq B\|x\|, \forall, x \in X.$$

And a Banach frame with respect to the Banach space $l^p(\Lambda)$ for X is a $p-$frame for X with a reconstruction operator S(see[1,13]).

In this paper, a new notion of frames is introduced: operator frames. We will see that an operator frame is different from a frame for Hilbert spaces and a frame of subspaces. When we consider an operator frame as a sequence of elements in the dual space of space $L^1(\mathcal{H})$ of trace-class operators on a Hilbert space $\mathcal{H}$ (since $B(\mathcal{H}) = L^1(\mathcal{H})^*$(see[15])), it is not difficult to find out that an operator frame is also different from a Banach frame for $L^1(\mathcal{H})$. In Section 2, we will give the concepts of operator Bessel sequences, operator frames and operator Riesz bases and discuss some properties of them. It is well-known that if a Banach frame $\{T_i\}_{i\in\Lambda}$ for $L^1(\mathcal{H})$, has a dual, we can reconstruct every element in $L^1(\mathcal{H})$ and its dual space $B(\mathcal{H})$ immediately. Unluckily, a dual of a Banach frame does not always exist and so this reconstruction maybe not be realized. One can refer to References [1,4,5,7,13] for the theory of Banach frames. In Section 3, dual frames of an operator frame is defined. The advantages of an operator frame are shown by the following facts. First, a dual of an operator frame always exists; second, we can reconstruct not only elements of $\mathcal{H}$, but also elements of $B(\mathcal{H})$ pointwisely by using an operator frame and its dual frame. In order to establish a relationship between operator frames and frames for Hilbert spaces, we introduce the notion of operator response in Section 4. In this section, we shall see that, in some sense, the notion of operator frames can be viewed as a generalization of frames for Hilbert spaces.

2. Operator Frames

Definition 2.1. A family of bounded linear operators $\{T_i\}_{i\in\Lambda}$ on a complex Hilbert space $\mathcal{H}$ is an operator Bessel sequence in $B(\mathcal{H})$, if there exists a positive constant $M > 0$ such that

$$\sum_{i\in\Lambda} \|T_i x\|^2 \leq M\|x\|^2, \quad \forall x \in \mathcal{H},$$

M is called a bound for $\{T_i\}_{i\in\Lambda}$. Denote by $B_{\mathcal{H}}$ the set of all operator Bessel sequences in $B(\mathcal{H})\}$ indexed by Λ.

For any elements $T = \{T_i\}_{i\in\Lambda}$ and $S = \{S_i\}_{i\in\Lambda}$ of $B_{\mathcal{H}}$, we define

$$\lambda T + \mu S = \{\lambda T_i + \mu S_i\}_{i\in\Lambda}, \quad \forall \lambda, \mu \in \mathbb{C},$$

and

$$\|T\| = \sup_{\|x\|\leq 1} (\sum_{i\in\Lambda} \|T_i x\|^2)^{\frac{1}{2}}.$$

We can easily check that $(B_{\mathcal{H}}, \|\cdot\|)$ is a Banach space.

Let $\mathcal{H}$ be a Hilbert space. We denote that

$$S(\mathcal{H}) = \{\{x_i\}_{i\in\Lambda} : x_i \in \mathcal{H}\}$$

$$l^2(\mathcal{H}) = \{\{x_i\}_{i\in\Lambda} : x_i \in \mathcal{H}, \sum_{i\in\Lambda} \|x_i\|^2 < \infty\}.$$

and define an inner product

$$\langle \{x_i\}_{i\in\Lambda}, \{y_i\}_{i\in\Lambda} \rangle = \sum_{i\in\Lambda} \langle x_i, y_i \rangle.$$

Proposition 6.2 in [8] yields that $l^2(\mathcal{H})$ is a Hilbert space.

Let $T = \{T_i\}_{i\in\Lambda}$ be an operator Bessel sequence in $B(\mathcal{H})$ and the operator

$$R_T : \mathcal{H} \to l^2(\mathcal{H})$$

be defined by

$$R_T x = \{T_i x\}_{i\in\Lambda}, \quad \forall x \in \mathcal{H}.$$

Clearly, R_T is a bounded linear operator.

Define

$$\alpha(T) = R_T, \forall T \in B_{\mathcal{H}},$$

then we get a mapping α from $(B_{\mathcal{H}}, \|\cdot\|)$ to $B(\mathcal{H}, l^2(\mathcal{H}))$. It is clear that α is linear and isometric.

Let $\{e_i\}_{i\in\Lambda}$ be an orthonormal basis for $\mathcal{H}$ and A be an operator in $B(\mathcal{H}, l^2(\mathcal{H}))$. Assume that

$$Ae_i = \{f_j^i\}_{j\in\Lambda} \in l^2(\mathcal{H}), \forall i \in \Lambda,$$

then

$$Ax = A(\sum_{i\in\Lambda} c_i e_i) = \{\sum_{i\in\Lambda} c_i f_j^i\}_{j\in\Lambda} \in l^2(\mathcal{H}), \forall x = \sum_{i\in\Lambda} c_i e_i \in \mathcal{H}.$$

This shows that for any $x = \sum_{i\in\Lambda} c_i e_i$ in $\mathcal{H}$, $\sum_{i\in\Lambda} c_i f_j^i$ converges. Define a sequence of operators $T = \{T_j\}_{j\in\Lambda}$ by

$$T_j x = \sum_{i\in\Lambda} c_i f_j^i, \forall x = \sum_{i\in\Lambda} c_i e_i \in \mathcal{H}.$$

From [9, Theorem 1], we have that there exists $M > 0$ such that

$$\|T_j x\|^2 = \|\sum_{i\in\Lambda} c_i f_j^i\|^2 \leq M\|x\|^2.$$

Thus, $\{T_j\}_{j\in\Lambda} \subset B(\mathcal{H})$. For any $x = \sum_{i\in\Lambda} c_i e_i \in \mathcal{H}$, we get

$$\sum_{j\in\Lambda} \|T_j x\|^2 = \sum_{j\in\Lambda} \|\sum_{i\in\Lambda} c_i f_j^i\|^2 = \|Ax\|^2 \leq \|A\|^2\|x\|^2.$$

Hence, $T = \{T_j\}_{j\in\Lambda}$ is an operator Bessel sequence. Compute

$$R_T x = \{T_j x\}_{j\in\Lambda} = \{\sum_{i\in\Lambda} c_i f_j^i\}_{j\in\Lambda} = Ax, \forall x = \sum_{i\in\Lambda} c_i e_i \in \mathcal{H}.$$

This shows that $R_T = A$. Consequently, the space $B_{\mathcal{H}}$ and the operator space $B(\mathcal{H}, l^2(\mathcal{H}))$ are isometrically isomorphic. This fact will be used in our discussion on operator frames.

Definition 2.2. Let $T = \{T_i\}_{i\in\Lambda}$ be an operator Bessel sequence in $B(\mathcal{H})$, then the operator R_T is called the analysis operator of $T = \{T_i\}_{i\in\Lambda}$.

Lemma 2.3. *Let $T = \{T_i\}_{i\in\Lambda}$ be an operator Bessel sequence in $B(\mathcal{H})$. Then the adjoint of R_T is given by*

$$R_T^*(\{x_i\}_{i\in\Lambda}) = \sum_{i\in\Lambda} T_i^* x_i, \ \forall \{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H}).$$

Proof. Assume that $T = \{T_i\}_{i\in\Lambda}$ is an operator Bessel sequence in $B(\mathcal{H})$ with a bound M_T. $\{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H})$ means that $\sum_{i\in\Lambda} \|x_i\|^2$ converges i.e. $\forall \varepsilon > 0, \exists N \in \mathcal{F}(\Lambda)$ such that $\sum_{i\in L} \|x_i\|^2 < \varepsilon$, whenever $L \in \mathcal{F}(\Lambda)$ with $L \cap N = \emptyset$. For every $L \in \mathcal{F}(\Lambda)$ with $L \cap N = \emptyset$, we can find a unit vector $x_0 \in \mathcal{H}$ such that

$$\|\sum_{i\in L} T_i^* x_i\| = |\langle \sum_{i\in L} T_i^* x_i, x_0\rangle| = |\sum_{i\in L} \langle x_i, T_i x_0\rangle|.$$

Hence,

$$\begin{aligned} \|\sum_{i\in L} T_i^* x_i\|^2 &= |\sum_{i\in L} \langle x_i, T_i x_0\rangle|^2 \\ &\leq \sum_{i\in L} \|x_i\|^2 \cdot \sum_{i\in\Lambda} \|T_i x_0\|^2 \\ &\leq M_T \varepsilon. \end{aligned}$$

This shows that $\sum_{i\in\Lambda} T_i^* x_i$ converges. Thus, R_T^* is well-defined. Also, it is easy to check that the adjoint of R_T is defined as above. $\square$

We call R_T^* the synthesis operator of $T = \{T_i\}_{i\in\Lambda}$.

Theorem 2.4. *Let $T = \{T_i\}_{i\in\Lambda}$ be a sequence of operators on $\mathcal{H}$, then the following statements are equivalent.*

(1) *$T = \{T_i\}_{i\in\Lambda}$ is an operator Bessel sequence in $B(\mathcal{H})$.*
(2) *There exists a positive constant M such that*
$$\|\sum_{i\in\Lambda} T_i^* x_i\|^2 \leq M\|\{x_i\}_{i\in\Lambda}\|^2, \quad \forall \{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H}).$$
(3) *There exists a positive constant M such that*
$$\|\sum_{i\in L} T_i^* x_i\|^2 \leq M\sum_{i\in L}\|x_i\|^2, \quad \forall \{x_i\}_{i\in\Lambda} \in S(\mathcal{H}), \forall L \in \mathcal{F}(\Lambda).$$
(4) *For every $\{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H})$, $\sum_{i\in\Lambda} T_i^* x_i$ converges.*

Proof. Clearly, (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is always true.

Next, we prove (1) $\Leftrightarrow$ (2). Assume that $T = \{T_i\}_{i\in\Lambda}$ is an operator Bessel sequence in $B(\mathcal{H})$ with analysis operator R_T. Put $M = \|R_T^*\|^2$, then
$$\begin{aligned}\|\sum_{i\in\Lambda} T_i^* x_i\|^2 &= \|R_T^*(\{x_i\}_{i\in\Lambda})\|^2 \\ &\leq \|R_T^*\|^2 \cdot \|\{x_i\}_{i\in\Lambda})\|^2 \\ &= M\|\{x_i\}_{i\in\Lambda})\|^2, \qquad \forall \{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H}).\end{aligned}$$
Conversely, if (2) holds, we define an bounded linear operator $Q : l^2(\mathcal{H}) \to \mathcal{H}$ by $Q(\{x_i\}_{i\in\Lambda}) = \sum_{i\in\Lambda} T_i^* x_i$, and compute that
$$Q^* x = \{T_i x\}_{i\in\Lambda} \in l^2(\mathcal{H}), \quad \forall x \in \mathcal{H}.$$
Hence,
$$\sum_{i\in\Lambda}\|T_i x\|^2 = \|\{T_i x\}_{i\in\Lambda}\|^2 = \|Q^* x\|^2 \leq \|Q^*\|^2 \cdot \|x\|, \quad \forall x \in \mathcal{H}.$$
So $T = \{T_i\}_{i\in\Lambda}$ is an operator Bessel sequence in $B(\mathcal{H})$. $\square$

Definition 2.5. A family of bounded linear operators $\{T_i\}_{i\in\Lambda}$ on a Hilbert space $\mathcal{H}$ is said to be an operator frame for $B(\mathcal{H})$, if there exist positive constants $A, B > 0$ such that
$$A\|x\|^2 \leq \sum_{i\in\Lambda}\|T_i x\|^2 \leq B\|x\|^2, \forall x \in \mathcal{H}.$$
Where A and B are called a lower bound and an upper bound for the operator frame, respectively. An operator frame $\{T_i\}_{i\in\Lambda}$ is said to be tight if the constants A and B can be chosen equally. It is called a Parseval operator frame if $A = B = 1$, and a self-adjoint operator frame if every operator T_i is self-adjoint, i.e. $T_i = T_i^*$.

Let $\{W_i\}_{i\in\Lambda}$ be a frame of subspaces with respect to $\{v_i\}_{i\in\Lambda}$ for $\mathcal{H}$. Put $T_i = v_i\pi_{W_i}, \forall i \in \Lambda$, then we get a sequence of operators $\{T_i\}_{i\in\Lambda}$. Clearly, the condition (1.1) implies that there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in\Lambda} \|T_i x\|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}.$$

Thus, the sequence $\{T_i\}_{i\in\Lambda}$ becomes an operator frame for $\mathcal{H}$. With this point of view, a frame of subspaces can be viewed as a special case of operator frames. Another relationship between operator frames and frames of subspaces is given in Theorem 3.6 below.

Example 2.6. Let $L^2(\mathbb{R})$ be the Hilbert space consisting of all Lebesgue measurable and square integrable functions on $\mathbb{R}$. For every $f \in L^2(\mathbb{R})$, we define operators V, U in $B(L^2(\mathbb{R}))$ in such a way that

$$(Vf)(x) = f(x-1), \ (Uf)(x) = 2^{\frac{1}{4}} f(2x).$$

Then for every fixed $k \in \mathbb{Z}$, the sequence $\{U^j V^k\}_{j\in\mathbb{N}}$ is a tight operator frame for $L^2(\mathbb{R})$.

Indeed, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\sum_{j\in\mathbb{N}} \|U^j V^k f\|^2 &= \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} |U^j V^k f(x)|^2 dx \\
&= \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} |2^{\frac{j}{4}} f(2^j x - k)|^2 dx \\
&= \sum_{j\in\mathbb{N}} 2^{\frac{j}{2}} \int_{\mathbb{R}} |f(2^j x - k)|^2 dx \\
&= \sum_{j\in\mathbb{N}} 2^{-\frac{j}{2}} \int_{\mathbb{R}} |f(y)|^2 dy (y = 2^j x) \\
&= \sum_{j\in\mathbb{N}} 2^{-\frac{j}{2}} \|f\|^2 \\
&= \|f\|^2 \sum_{j\in\mathbb{N}} 2^{-\frac{j}{2}} \\
&= (2 + \sqrt{2})\|f\|^2.
\end{aligned}$$

Theorem 2.7. *Assume that $\{T_i\}_{i\in\Lambda}$ is an operator Bessel sequence in $B(\mathcal{H})$. If $\{T_i\}_{i\in\Lambda}$ is a 2−frame for $L^1(\mathcal{H})$, then it is an operator frame for $B(\mathcal{H})$.*

Proof. Since $\{T_i\}_{i\in\Lambda}$ is a 2−frame for $L^1(\mathcal{H})$, there are constants $A, B > 0$ such that

$$A\|x \otimes x\|_1^2 \leq \sum_{i\in\Lambda} |\langle T_i, x \otimes x\rangle|^2 \leq B\|x \otimes x\|_1^2, \quad \forall x \in \mathcal{H}.$$

From the theory of trace-class operators, we can get

$$\begin{aligned}\sum_{i\in\Lambda}|\langle T_i, x\otimes x\rangle|^2 &= \sum_{i\in\Lambda}|\mathrm{tr}((x\otimes x)T_i)|^2\\ &= \sum_{i\in\Lambda}|\mathrm{tr}(x\otimes T_i^*x)|^2\\ &= \sum_{i\in\Lambda}|\langle T_i x, x\rangle|^2\\ &\le \sum_{i\in\Lambda}\|T_i x\|^2\|x\|^2.\end{aligned}$$

Note that $\|x\otimes y\|_1 = \|x\|\|y\|$, we obtain that

$$A\|x\|^2 \le \sum_{i\in\Lambda}\|T_i x\|^2.$$

Since $\{T_i\}_{i\in\Lambda}$ is an operator Bessel sequence, it is an operator frame for $B(\mathcal{H})$. □

Assume that $T = \{T_i\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$ and R_T, R_T^* are the analysis operator and synthesis operator of $T = \{T_i\}_{i\in\Lambda}$, respectively, we define the frame operator S_T of $T = \{T_i\}_{i\in\Lambda}$ as $S_T = R_T^* R_T$.

Theorem 2.8. *Assume that S_T is the frame operator of an operator frame $T = \{T_i\}_{i\in\Lambda}$ for $B(\mathcal{H})$ with bounds A, B, then S_T is a positive invertible operator on $\mathcal{H}$ and $AI \le S_T \le BI$. Moreover, we have a reconstruction formula*

$$x = \sum_{i\in\Lambda} S_T^{-1}T_i^*T_i x, \quad \forall x\in\mathcal{H}.$$

Proof. It is clear that S_T is positive. For any $x\in\mathcal{H}$, since $T = \{T_i\}_{i\in\Lambda}$ is an operator frame with bounds A, B, we have

$$\langle Ax, x\rangle = A\|x\|^2 \le \sum_{i\in\Lambda}\|T_i x\|^2 = \langle S_T x, x\rangle \le \langle Bx, x\rangle.$$

This shows that

$$AI \le S_T \le BI,$$

which implies that S_T is invertible. Further, for any $x\in\mathcal{H}$, we have

$$x = S_T^{-1}S_T x = S_T^{-1}\sum_{i\in\Lambda}T_i^*T_i x = \sum_{i\in\Lambda}S_T^{-1}T_i^*T_i x.$$

□

Definition 2.9. A sequence $T = \{T_i\}_{i\in\Lambda}$ in $B(\mathcal{H})$ is said to be independent, if the following condition is satisfied:

$$\sum_{i\in\Lambda}T_i^*x_i = 0, \{x_i\}_{i\in\Lambda}\in S(\mathcal{H}) \Longrightarrow x_i = 0, \forall i\in\Lambda.$$

Theorem 2.10. *Let $T = \{T_i\}_{i\in\Lambda}$ be an operator Bessel sequence in $B(\mathcal{H})$, then*

(1) *$T = \{T_i\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$ if and only if R_T is below bounded.*
(2) *$T = \{T_i\}_{i\in\Lambda}$ is an independent operator frame for $B(\mathcal{H})$ if and only if R_T is invertible.*

Proof. The proof of (1) is easy, so we omit it. Assume that $T = \{T_i\}_{i\in\Lambda}$ is independent, we now prove that R_T is invertible. On the one hand, from the condition and the definition of independent operator frame, we can know that R_T^* is injective, and so $R(R_T) = \ker(R_T^*) = \{0\}$. This shows that range of R_T is dense in $\mathcal{H}$. On the other hand, from (1), we know that R_T is below bounded and so $R(R_T)$ is closed. Hence R_T is invertible. Conversely, if R_T is invertible, then R_T is below bounded. Thus, $T = \{T_i\}_{i\in\Lambda}$ is an operator frame. Now, suppose that $T = \{T_i\}_{i\in\Lambda}$ is not independent, then there exist a non-zero sequence $\{x_i\}_{i\in\Lambda} \subset \mathcal{H}$ and some $i_0 \in \Lambda$ such that $x_{i_0} \neq 0$. Thus

$$T_{i_0}^* x_{i_0} = \sum_{i\neq i_0} T_i^* x_i. \tag{2.1}$$

Since R_T is also surjective, there exists $x \in \mathcal{H}$ such that $R_T x = \{T_j x\}_{j\in\Lambda} = \eta_{i_0} \in l^2(\mathcal{H})$, where $\eta_{i_0} = \{y_j\}_{j\in\Lambda}, y_{i_0} = x_{i_0}$ and $y_j = 0, j \neq i_0$. Hence, $T_{i_0} x = x_{i_0}$ and so $\langle x, T_{i_0}^* x_{i_0}\rangle = \langle T_{i_0} x, x_{i_0}\rangle = \|x_{i_0}\|^2 \neq 0$. But (2.1) can implies that

$$\langle x, T_{i_0}^* x_{i_0}\rangle = \langle x, \sum_{i\neq i_0} T_i^* x_i\rangle = \sum_{i\neq i_0}\langle x, T_i^* x_i\rangle = \sum_{i\neq i_0}\langle T_i x, x_i\rangle = 0.$$

a contradiction. This shows that $T = \{T_i\}_{i\in\Lambda}$ is independent. □

Definition 2.11. A family of operators $\{T_i\}_{i\in\Lambda}$ on $\mathcal{H}$ is called, if

$$\overline{\mathfrak{span}}\{T_i^*\}_{i\in\Lambda} = \mathcal{H},$$

where

$$\overline{\mathfrak{span}}\{T_i^*\}_{i\in\Lambda} = \text{the cloure of } \{\sum_{i\in L} T_i^* x_i : \{x_i\}_{i\in\Lambda} \in S(\mathcal{H}), \forall L \in \mathcal{F}(\Lambda)\}.$$

Definition 2.12. A operator sequence $T = \{T_i\}_{i\in\Lambda}$ on $\mathcal{H}$ is an operator Riesz basis for $B(\mathcal{H})$, if it satisfies:

1. $\overline{\mathfrak{span}}\{T_i^*\}_{i\in\Lambda} = \mathcal{H}$ and
2. there exist constants $C, D > 0$ such that

$$C\|\{x_i\}_{i\in\Lambda}\|^2 \leq \|\sum_{i\in\Lambda} T_i^* x_i\|^2 \leq D\|\{x_i\}_{i\in\Lambda}\|^2, \quad \forall\{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H}). \tag{2.2}$$

Immediately, we know that every operator Riesz basis is always an operator frame from the following conclusion.

Theorem 2.13. *Let $T = \{T_i\}_{i\in\Lambda}$ be a sequence of operators on $\mathcal{H}$, then the following statements are equivalent.*

(1) *$T = \{T_i\}_{i\in\Lambda}$is an operator Riesz basis.*
(2) *$T = \{T_i\}_{i\in\Lambda}$ is an independent operator frame.*

Proof. If $T = \{T_i\}_{i\in\Lambda}$ is an operator Riesz basis, then R_T^* is below bounded by Theorem 2.10 and so the range of R_T^* is closed and denote $R(R_T^*)$. In addition, $\overline{\mathfrak{span}}\{T_i^*\}_{i\in\Lambda} = R(R_T^*) = \mathcal{H}$. Thus, R_T^* is bijective. Applying the Banach Inverse Theorem, R_T^* is invertible and so R_T is also invertible. Hence, Theorem 2.4(2) and Theorem 2.10(2) imply that $T = \{T_i\}_{i\in\Lambda}$ is an independent operator frame.

Conversely, assume that $T = \{T_i\}_{i\in\Lambda}$ is an independent operator frame, then Theorem 2.10(2) shows that R_T is invertible. Thus R_T^* is invertible. For any $\{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H})$, we have

$$\|\{x_i\}_{i\in\Lambda}\|^2 = \|R_T^{*-1}R_T^*(\{x_i\}_{i\in\Lambda})\|^2 \le \|R_T^{*-1}\|^2\|R_T^*(\{x_i\}_{i\in\Lambda})\|^2.$$

Put $C = \|R_T^{*-1}\|^{-2}, D = \|R_T^*\|^2$, then

$$\begin{aligned} C\|\{x_i\}_{i\in\Lambda}\|^2 &\le \|R_T^*(\{x_i\}_{i\in\Lambda})\|^2 = \|\sum_{i\in\Lambda} T_i^* x_i\|^2 \\ &\le D\|\{x_i\}_{i\in\Lambda}\|^2, \quad \forall \{x_i\}_{i\in\Lambda} \in l^2(\mathcal{H}). \end{aligned}$$

Thus, condition (2.2) holds. Since R_T^* is invertible, the condition (1) of Definition 2.12 holds. Hence $T = \{T_i\}_{i\in\Lambda}$ is an operator Riesz basis. □

3. Dual of Operator Frames

In this section, we will study reconstruction problem. Thus we need introduce the notion of dual frames of an operator frame.

Definition 3.1. Let $T = \{T_i\}_{i\in\Lambda}$ be an operator frame for $B(\mathcal{H})$. A family of operators $\widetilde{T} = \{\widetilde{T}_i\}_{i\in\Lambda}$ on $\mathcal{H}$ is called a dual of operator frame $T = \{T_i\}_{i\in\Lambda}$ if they satisfy

$$x = \sum_{i\in\Lambda} T_i^*\widetilde{T}_i x \quad \forall x \in \mathcal{H}. \tag{3.1}$$

Furthermore, we call $\{\widetilde{T}_i\}_{i\in\Lambda}$ a dual frame of the operator frame $T = \{T_i\}_{i\in\Lambda}$ if $\{\widetilde{T}_i\}_{i\in\Lambda}$ is also an operator frame for $B(\mathcal{H})$ and satisfies condition (3.1).

Theorem 3.2. *Every operator frame for $B(\mathcal{H})$ has a dual frame.*

Proof. If $T = \{T_i\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$ with bounds A, B, then the operator sequence $\widetilde{T} = \{T_i S_T^{-1}\}_{i\in\Lambda}$ is a dual frame of $T = \{T_i\}_{i\in\Lambda}$. In fact, we have

$$x = S_T S_T^{-1}x = \sum_{i\in\Lambda} T_i^* T_i S_T^{-1} x = \sum_{i\in\Lambda} T_i^*\widetilde{T}_i x, \quad \forall x \in \mathcal{H}.$$

And $\widetilde{T} = \{T_i S_T^{-1}\}_{i\in\Lambda}$ satisfies

$$A\|S_T\|^{-2}\cdot\|x\|^2 \le \sum_{i\in\Lambda}\|\widetilde{T}_i x\|^2 = \sum_{i\in\Lambda}\|T_i S_T^{-1}x\|^2 \le B\|S_T^{-1}\|^2\cdot\|x\|^2, \quad \forall x \in \mathcal{H}.$$

Hence, $\{T_i S_T^{-1}\}_{i\in\Lambda}$ is called the canonical dual frame of $\{T_i\}_{i\in\Lambda}$. □

Remark 3.3. Assume that $T = \{T_i\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$ with analytic operator R_T and $\widetilde{T} = \{\widetilde{T}_i\}_{i\in\Lambda}$ is a dual frame of T with analytic operator $R_{\widetilde{T}}$, Then for any x in $\mathcal{H}$, we have

$$x = \sum_{i\in\Lambda} T_i^* \widetilde{T}_i x = R_T^* R_{\widetilde{T}} x. \tag{3.2}$$

This shows that every element of $\mathcal{H}$ can be reconstructed with an operator frame for $B(\mathcal{H})$ and its dual frame.

Moreover, we also have another fact that for any operator A on $\mathcal{H}$, we get

$$Ax = \sum_{i\in\Lambda} T_i^* \widetilde{T}_i Ax, \quad \forall x \in \mathcal{H}. \tag{3.3}$$

That is, an association of operator frame and its dual frame can reconstruct pointwisely every operator on $\mathcal{H}$ and so we can write

$$A \doteq \sum_{i\in\Lambda} T_i^* \widetilde{T}_i A,$$

where $\sum\limits_{i\in\Lambda} T_i^* \widetilde{T}_i A$ converges strongly to A.

Lemma 3.4. *Let $T = \{T_i\}_{i\in\Lambda}$ be an operator frame for $B(\mathcal{H})$ with bounds A, B. If $Q = \{Q_i\}_{i\in\Lambda}$ is an operator Bessel sequence in $B(\mathcal{H})$ with a bound $M < \frac{A^2}{4B}$, then $T \pm Q = \{T_i \pm Q_i\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$.*

Proof. We only prove the case that $T+Q = \{T_i+Q_i\}_{i\in\Lambda}$. The other case is similar.

For any $x \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in\Lambda} \|(T_i + Q_i)x\|^2 &\le \sum_{i\in\Lambda} (\|T_i x\| + \|Q_i x\|)^2 \\
&= \sum_{i\in\Lambda} \|T_i x\|^2 + \sum_{i\in\Lambda} \|Q_i x\|^2 + 2\sum_{i\in\Lambda} \|T_i x\| \|Q_i x\| \\
&\le B\|x\|^2 + M\|x\|^2 + 2(\sum_{i\in\Lambda} \|T_i x\|^2)^{\frac{1}{2}} \cdot (\sum_{i\in\Lambda} \|Q_i x\|^2)^{\frac{1}{2}} \\
&\le (B + M)\|x\|^2 + 2\sqrt{B}\sqrt{M}\|x\|^2 \\
&\le (B + M + 2\sqrt{B}\sqrt{M})\|x\|^2.
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i\in\Lambda} \|(T_i + Q_i)x\|^2 &\ge \sum_{i\in\Lambda} (\|T_i x\| - \|Q_i x\|)^2 \\
&= \sum_{i\in\Lambda} \|T_i x\|^2 + \sum_{i\in\Lambda} \|Q_i x\|^2 - 2\sum_{i\in\Lambda} \|T_i x\| \|Q_i x\| \\
&\ge A\|x\|^2 + \sum_{i\in\Lambda} \|Q_i x\|^2 - 2(\sum_{i\in\Lambda} \|T_i x\|^2)^{\frac{1}{2}} \cdot (\sum_{i\in\Lambda} \|Q_i x\|^2)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\geq \quad A\|x\|^2 - 2(\sum_{i\in\Lambda}\|T_ix\|^2)^{\frac{1}{2}} \cdot (\sum_{i\in\Lambda}\|Q_ix\|^2)^{\frac{1}{2}} \\
&\geq \quad (A - 2\sqrt{B}\sqrt{M})\|x\|^2.
\end{aligned}$$

Hence, $T + Q = \{T_i + Q_i\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$. □

Theorem 3.5. *Let $T = \{T_i\}_{i\in\Lambda}$ be an operator frame for $B(\mathcal{H})$ with bounds A, B, then the following statements are equivalent.*

(1) $T = \{T_i\}_{i\in\Lambda}$ *is independent.*
(2) $T = \{T_i\}_{i\in\Lambda}$ *is an operator Riesz basis.*
(3) $R(R_T) = l^2(\mathcal{H})$.
(4) $T = \{T_i\}_{i\in\Lambda}$ *has a unique dual frame.*

Proof. Theorem 2.10 and Theorem 2.13 yield that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). It need only prove (1) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (4) Suppose that independent operator frame $\{T_i\}_{i\in\Lambda}$ have two dual frames: $\widetilde{T} = \{\widetilde{T}_i\}_{i\in\Lambda}$ and $\widetilde{Q} = \{\widetilde{Q}_i\}_{i\in\Lambda}$, then $R_{\widetilde{T}}$ and $R_{\widetilde{Q}}$ are left inverses of R_T^* from (3.2). Thus $\widetilde{T} = \widetilde{Q}$.

(4) $\Rightarrow$ (1) Assume that $T = \{T_i\}_{i\in\Lambda}$ has a unique dual frame $\widetilde{T} = \{\widetilde{T}_i\}_{i\in\Lambda}$. Suppose that $T = \{T_i\}_{i\in\Lambda}$ is not independent, then $R(R_T) \neq l^2(\mathcal{H})$, i.e. $R(R_T)^\perp \neq \{0\}$. Thus there exists a non-zero element $\{x_i\}_{i\in\Lambda} \in R(R_T)^\perp$ such that $\|\{x_i\}_{i\in\Lambda}\| < \frac{A}{2\sqrt{B}}$. Take a unit vector $e \in \mathcal{H}$, define a sequence of bounded linear operators $\widetilde{U} = \{\widetilde{U}_i\}_{i\in\Lambda}$ in such a way that $\widetilde{U}_ix = \langle x, e\rangle x_i, \forall x \in \mathcal{H}$. Put $\widetilde{Q} = \widetilde{U} + \widetilde{T}$, then for all x in $\mathcal{H}$,

$$\begin{aligned}
\sum_{i\in\text{Ł}}\|\widetilde{U}_ix\|^2 \quad &= \quad \sum_{i\in\Lambda}\|\langle x, e\rangle x_i\|^2 \leq \sum_{i\in\Lambda}\|\langle x, e\rangle\|^2\|x_i\|^2 \\
&\leq \quad \|x\|^2\sum_{i\in\Lambda}\|x_i\|^2.
\end{aligned}$$

Thus, the sequence $\{\widetilde{U}_i\}_{i\in\Lambda}$ is an operator Bessel sequence with a Bessel bound less than $\frac{A^2}{4B}$. By Lemma 3.4 we know that $\widetilde{Q}$ is an operator frame for $B(\mathcal{H})$. For any $x \in \mathcal{H}$, since $\{\widetilde{U}_ix\}_{i\in\Lambda} = \{\langle x, e\rangle x_i\}_{i\in\Lambda} \in R(R_T)^\perp = \ker(R_T^*)$, we see that $R_T^*(\{\widetilde{U}_ix\}_{i\in\Lambda}) = \sum\limits_{i\in\Lambda} T_i^*\widetilde{U}_ix = 0$, and so

$$x = \sum_{i\in\Lambda} T_i^*\widetilde{T}_ix = \sum_{i\in\Lambda} T_i^*(\widetilde{T}_i + \widetilde{U}_i)x = \sum_{i\in\Lambda} T_i^*\widetilde{Q}_ix.$$

Thus $\widetilde{Q}$ is also a dual frame of $T = \{T_i\}_{i\in\Lambda}$. Clearly, $\widetilde{Q} \neq \widetilde{T}$. This contradicts the uniqueness of the dual frame of T. Hence, $T = \{T_i\}_{i\in\Lambda}$ is independent. □

For a frame of subspaces $\{W_i\}_{i\in\Lambda}$ with respect to the family of weights $\{v_i\}_{i\in\Lambda}$ for $\mathcal{H}$ with synthesis operator $T_{W,v}$, the sequence $\{u_i\}_{i\in\Lambda} = \{S_{W,v}^{-1}W_i\}_{i\in\Lambda}$ is called the dual frame of $\{W_i\}_{i\in\Lambda}$(see[3]), where the operator $S_{W,v} = T_{W,v}T_{W,v}^*$.

Theorem 3.6. *For a frame of subspaces $\{W_i\}_{i\in\Lambda}$ with respect to the family of weights $\{v_i\}_{i\in\Lambda}$ for $\mathcal{H}$, define $T_i = v_i S_{W,v}\pi_{W_i} S^{-1}_{W,v}$ and $Q_i = v_i \pi_{u_i} S^{-1}_{W,v}$, then $Q = \{Q_i\}_{i\in\Lambda}$ and $T = \{T_i\}_{i\in\Lambda}$ are all operator frames for $B(\mathcal{H})$, and Q is a dual frame of T.*

Proof. Assume that $\{W_i\}_{i\in\Lambda}$ has frame bounds A, B.

Claim 1. $T = \{T_i\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$.

For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in\Lambda}\|T_i x\|^2 &= \sum_{i\in\Lambda}\|v_i S_{W,v}\pi_{W_i}S^{-1}_{W,v}x\|^2\\ &\le \|S_{W,v}\|^2 B\|S^{-1}_{W,v}x\|^2\\ &\le B\|S_{W,v}\|^2\|S^{-1}_{W,v}\|^2\|x\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i\in\Lambda}\|T_i x\|^2 &= \sum_{i\in\Lambda}\|v_i S_{W,v}\pi_{W_i}S^{-1}_{W,v}x\|^2\\ &\ge \sum_{i\in\Lambda}\|S^{-1}_{W,v}\|^{-2}\|v_i\pi_{W_i}S^{-1}_{W,v}x\|^2\\ &\ge \|S^{-1}_{W,v}\|^{-2}A\|S^{-1}_{W,v}x\|^2\\ &\ge A\|S^{-1}_{W,v}\|^{-2}\|S_{W,v}\|^{-2}\|x\|^2.\end{aligned}$$

Thus $T = \{T_i\}_{i\in\Lambda}$ is an operator frame.

Claim 2. $Q = \{Q_i\}_{i\in\Lambda}$ is also an operator frame for $B(\mathcal{H})$. The proof is similar to Claim 1.

Claim 3. $T_{U,v} = S^{-1}_{W,v}T_{W,v}S_{W,v}, T^*_{U,v} = S^{-1}_{W,v}T^*_{W,v}S_{W,v}, S_{U,v} = S_{W,v}$.

It is easy to check that $\pi_{u_i} = S^{-1}_{W,v}\pi_{W_i}S_{W,v}$. Thus $T_{U,v} = S^{-1}_{W,v}T_{W,v}S_{W,v}$, $T^*_{U,v} = S^{-1}_{W,v}T^*_{W,v}S_{W,v}$ and so

$$\begin{aligned}S_{U,v} &= T_{U,v}T^*_{U,v}\\ &= S^{-1}_{W,v}T_{W,v}S_{W,v}S^{-1}_{W,v}T^*_{W,v}S_{W,v}\\ &= S^{-1}_{W,v}T_{W,v}T^*_{W,v}S_{W,v}\\ &= S^{-1}_{W,v}S_{W,v}S_{W,v}\\ &= S_{W,v}.\end{aligned}$$

Hence, for any $x \in \mathcal{H}$, we compute

$$\begin{aligned}\sum_{i\in\mathrm{L}}T^*_i Q_i x &= \sum_{i\in\Lambda}v_i S^{-1}_{W,v}\pi_{W_i}S_{W,v}\cdot v_i S^{-1}_{W,v}\pi_{W_i}S_{W,v}S^{-1}_{W,v}x\\ &= S^{-1}_{W,v}(\sum_{i\in\Lambda}v_i^2\pi_{W_i}x)\\ &= S^{-1}_{W,v}(S_{W,v}x)\\ &= x.\end{aligned}$$

This shows that $Q = \{Q_i\}_{i\in\Lambda}$ is an dual operator frame of the operator frame $T = \{T_i\}_{i\in\Lambda}$. □

In inequality (1.1), if $A = B$, we call $\{W_i\}_{i\in\Lambda}$ a Parseval frame of subspaces for $\mathcal{H}$.

Theorem 3.7. *Assume that $\{W_i\}_{i\in\Lambda}$ is a Parseval frame of subspaces for a Hilbert space $\mathcal{H}$, then $\{v_i\pi_{W_i}\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$ and a dual frame of itself.*

Proof. When $\{W_i\}_{i\in\Lambda}$ is a Parseval frame of subspaces, $S_{W,v} = I$. Clearly, this theorem is a consequence of Theorem 3.6. □

4. Operator Responses

In order to show that the notion of operator frames is a generalization of usual frames for Hilbert spaces, we introduce the concept of operator responses of elements of Hilbert spaces.

Definition 4.1. Let $\mathcal{H}$ be a Hilbert space and e be a unit vector (i.e. $\|e\| = 1$) in $\mathcal{H}$. For every f in $\mathcal{H}$, put $T_f^e = e \otimes f$, that is,

$$T_f^e x = \langle x, f\rangle e, \forall x \in \mathcal{H}.$$

It is well know that T_f^e is a linear bounded operator on $\mathcal{H}$ of rank ≤ 1 and $T_f^{e*} = f \otimes e$. We call T_f^e the operator response of f with respect to e. The set $\mathcal{R}_e^{\mathcal{H}} = \{T_f^e : f \in \mathcal{H}\}$ is called the operator response space of $\mathcal{H}$ with respect to e.

Theorem 4.2. *Assume that $\{f_i\}_{i\in\Lambda}$ is a sequence in $\mathcal{H}$ and $\{e_i\}_{i\in\Lambda}$ is a sequence of unit vectors in $\mathcal{H}$, then the following statements are valid.*

(1) *$\{f_i\}_{i\in\Lambda}$ is complete, i.e. $\overline{span}\{f_i : i \in \Lambda\} = \mathcal{H}$ if and only if $\{T_{f_i}^e\}_{i\in\Lambda}$ is complete.*
(2) *$\{f_i\}_{i\in\Lambda}$ is a frame for Hilbert space $\mathcal{H}$ if and only if $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is an operator frame for $B(\mathcal{H})$.*
(3) *$\{f_i\}_{i\in\Lambda}$ is a tight frame for Hilbert space $\mathcal{H}$ if and only if $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is a tight operator frame for $B(\mathcal{H})$.*
(4) *$\{f_i\}_{i\in\Lambda}$ is a normalized tight frame for Hilbert space $\mathcal{H}$ if and only if $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is a Parseval operator frame for $B(\mathcal{H})$.*
(5) *If $\{e_i\}$ is either not complete, or orthogonal, then $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is not independent.*

Proof. If $\{f_i\}_{i\in\Lambda}$ is complete, then $\forall x \in \mathcal{H}, \forall\varepsilon > 0$, there exits a sequence $\{c_i\}_{i\in\Lambda} \in \mathbb{C}$ and a finite set $L \in \mathcal{F}(\Lambda)$ such that

$$\|\sum_{i\in L} c_i f_i - x\| < \varepsilon.$$

Take $x_i = c_i e_i(\forall i \in \Lambda$ then $\langle x_i, e_i\rangle = c_i, \forall i \in \Lambda$ and so

$$\|\sum_{i\in L} T_{f_i}^{e_i*} x_i - x\| = \|\sum_{i\in L}\langle x_i, e_i\rangle f_i - x\| = \|\sum_{i\in L} c_i f_i - x\| < \varepsilon.$$

Thus $\overline{\mathfrak{span}}\{T_{f_i}^{e_i}\} = \mathcal{H}$, that is, $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is complete.

On the other hand, if $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is complete, then $\forall x \in \mathcal{H}, \forall \varepsilon > 0$, there exits a sequence $\{x_i\}_{i\in\Lambda} \in S(\mathcal{H})$ and a finite set $L \in \mathcal{F}(\Lambda)$ such that

$$\|\sum_{i\in L} T_{f_i}^{e_i\,*} x_i - x\| < \varepsilon.$$

That is,

$$\|\sum_{i\in L} \langle x_i, e_i\rangle f_i - x\| < \varepsilon.$$

Thus $\{f_i\}_{i\in\Lambda}$ is complete.

Moreover, for every $x \in \mathcal{H}$, we have

$$\sum_{i\in\Lambda} \|T_{f_i}^{e_i} x\|^2 = \sum_{i\in\Lambda} \|\langle x, f_i\rangle e_i\|^2 = \sum_{i\in\Lambda} |\langle x, f_i\rangle|^2. \tag{4.1}$$

Thus, (2) though (4) are valid.

Assume that $\{e_i\}$ is not complete, then $\{e_i : i \in \Lambda\}^\perp \neq \{0\}$. Take a nonzero sequence $\{x_i\}_{i\in\Lambda} \subset \{e_i : i \in \Lambda\}^\perp \setminus \{0\}$, then

$$\sum_{i\in\Lambda} T_{f_i}^{e_i\,*} x_i = \sum_{i\in\Lambda} \langle x_i, e_i\rangle f_i = 0. \tag{4.2}$$

This shows that the sequence $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is not independent. Next, we suppose that $\{e_i\}$ is orthogonal.Take a mapping $\phi : \Lambda \to \Lambda$ such that $\phi(i) \neq i$ for all $i \in \Lambda$ and define $x_i = e_{\phi(i)}$, then

$$\sum_{i\in\Lambda} T_{f_i}^{e_i\,*} x_i = \sum_{i\in\Lambda} \langle e_{\phi(i)}, e_i\rangle f_i = 0. \tag{4.3}$$

Hence, the sequence $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is not independent. $\square$

Theorem 4.3. *Let $\{f_i\}_{i\in\Lambda} \subset \mathcal{H}$, $\{\widetilde{f}_i\}_{i\in\Lambda} \subset \mathcal{H}$ and $\{e_i\}_{i\in\Lambda}$ be a sequence of unit vectors in $\mathcal{H}$, then the following statements are equivalent.*

(1) *$\{f_i\}_{i\in\Lambda}$ and $\{\widetilde{f}_i\}_{i\in\Lambda}$ are a pair of dual frames for $\mathcal{H}$.*
(2) *$\{T_{\widetilde{f}_i}^{e_i}\}_{i\in\Lambda}$ and $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ are dual frames of each other.*

Proof. (1) $\Longrightarrow$ (2) Let (1) hold. Then Theorem 4.2 implies that $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ and $\{T_{\widetilde{f}_i}^{e_i}\}_{i\in\Lambda}$ are operator frames for $B(\mathcal{H})$.

Second, for any $x \in \mathcal{H}$, we may compute

$$\begin{aligned}
\sum_{i\in\Lambda} T_{f_i}^{e_i\,*} T_{\widetilde{f}_i}^{e_i} x &= \sum_{i\in\Lambda} T_{f_i}^{e_i\,*} \langle x, \widetilde{f}_i\rangle e_i \\
&= \sum_{i\in\Lambda} \langle\langle x, \widetilde{f}_i\rangle e_i, e_i\rangle f_i \\
&= \sum_{i\in\Lambda} \langle x, \widetilde{f}_i\rangle\langle e_i, e_i\rangle f_i \\
&= \sum_{i\in\Lambda} \langle x, \widetilde{f}_i\rangle f_i \\
&= x.
\end{aligned}$$

Hence, $\{T_{\widetilde{f}_i}^{e_i}\}_{i\in\Lambda}$ is a dual frame of the operator frame $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$. Similarly, we can prove that $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ is a dual frame of the operator frame $\{T_{\widetilde{f}_i}^{e_i}\}_{i\in\Lambda}$.

(2) $\Longrightarrow$ (1) Suppose that $\{T_{\widetilde{f}_i}^{e_i}\}_{i\in\Lambda}$ and $\{T_{f_i}^{e_i}\}_{i\in\Lambda}$ are dual framed of each other, then we know from Theorem 4.2 that $\{f_i\}_{i\in\Lambda}$ and $\{\widetilde{f}_i\}_{i\in\Lambda}$ are frames for $\mathcal{H}$ and

$$\sum_{i\in\Lambda} T_{f_i}^{e_i\,*} T_{\widetilde{f}_i}^{e_i} x = x, \sum_{i\in\Lambda} T_{\widetilde{f}_i}^{e_i\,*} T_{f_i}^{e_i} x = x, \forall x \in \mathcal{H}.$$

Furthermore, for any $x \in \mathcal{H}$, we get

$$\begin{aligned}
x &= \sum_{i\in\Lambda} T_{f_i}^{e_i\,*} T_{\widetilde{f}_i}^{e_i} x = \sum_{i\in\Lambda} T_{f_i}^{e_i\,*} \langle x, \widetilde{f}_i\rangle e_i \\
&= \sum_{i\in\Lambda} \langle\langle x, \widetilde{f}_i\rangle e_i, e_i\rangle f_i = \sum_{i\in\Lambda} \langle x, \widetilde{f}_i\rangle\langle e_i, e_i\rangle f_i \\
&= \sum_{i\in\Lambda} \langle x, \widetilde{f}_i\rangle f_i
\end{aligned}$$

and

$$\begin{aligned}
x &= \sum_{i\in\Lambda} T_{\widetilde{f}_i}^{e_i\,*} T_{f_i}^{e_i} x = \sum_{i\in\Lambda} T_{\widetilde{f}_i}^{e_i\,*} \langle x, f_i\rangle e \\
&= \sum_{i\in\Lambda} \langle\langle x, f_i\rangle e_i, e_i\rangle \widetilde{f}_i = \sum_{i\in\Lambda} \langle x, f_i\rangle\langle e_i, e_i\rangle \widetilde{f}_i \\
&= \sum_{i\in\Lambda} \langle x, f_i\rangle \widetilde{f}_i.
\end{aligned}$$

Thus $\{f_i\}_{i\in\Lambda}$ and $\{\widetilde{f}_i\}_{i\in\Lambda}$ are a pair of dual frames for $\mathcal{H}$. □

From the discussion above, we can find that the concept of operator frames is a generalization of frames for Hilbert spaces. Operator responses of elements of a Hilbert space play an important role in studying the relation between operator frames and frames for Hilbert spaces.

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On the Stability of Multi-wavelet Frames

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Abstract. Frame plays an important role in the theory of wavelet analysis. Frame theory and stability of frames are important topics of wavelet analysis. Recently, people pay more attention to multi-wavelet frames. Among literatures, Chui[2], for instance, give a complete characterization of multi-wavelet frames for arbitrary dilation factor $a > 1$. There, however, is relatively less results on the stability of multi-wavelet frames. This paper devotes to the study of stability of multi-wavelet frames based on functional analysis methods. The following meaningful results are obtained: firstly, multi-wavelet frames are stable by some kinds of linear operators action; Secondly, multi-wavelet frames are stable over some kinds of perturbations conditions on ψ.

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1. Introduction

We begin with wavelet frames. Let $a > 1$, $b > 0$ and let $\psi_{j,k}(x) = a^{\frac{j}{2}}\psi(a^j x - bk)$, then when ψ, a, b satisfy given conditions, $\{\psi_{j,k}(x) : j,\, k \in Z\}$ constitute the *wavelet frames of* $L^2(R)$, that is, there exists constant A, $B > 0$ so that $A\|f\|^2 \le \sum\limits_{j,k\in Z} |\langle f,\ \psi_{j,\,k}(x)\rangle|^2 \le B\|f\|^2$, $\forall f \in L^2(R)$, where A and B are called *wavelet frame bounds*. However, for an arbitrary mother wavelet function $\Psi(x)$ and for an arbitrary dilation factor a and translation factor b, wavelet frames can not always be obtained. So, it is important to study the stability of wavelet frames. The wavelet frames $\{\psi_{j,\,k}(x)\}$ are called *stable*, when there is small perturbation on a, b, j, k or ψ, $\{\psi_{j,\,k}(x)\}$ are still wavelet frames. The stability of wavelet frames is needed in applications.

Many people pay attention to studying the stability of wavelet frames. In [5], Favier and Zalik study the stability of single-wavelet frame when there is perturbation conditions on k and ψ. In [1], Balan proves the stability of single-wavelet frame when there is perturbation conditions on b and ψ .

Recently, people pay more attention to multi-wavelet frames, such as: in [2], Chui studies multi-wavelet frames and establishes a complete characterization to multi-wavelet frames for an arbitrary dilation factor $a > 1$. But, there is relatively less results on the stability of multi-wavelet frames. In this paper, the stability of multi-wavelet frames are studied and some meaningful results are obtained.

2. Multi-wavelet Frames

Definition 2.1. [2] Let $1 < a < \infty$, and $b > 0$, A finite collection $\Psi_L(x) = \{\psi_1(x), \cdots, \psi_L(x)\}$ of functions in $L^2(R)$ is said to generate *an multi-wavelet frames*

$$F_a = \{\psi_{l,j,k}(x) = a^{\frac{j}{2}}\psi_l(a^j x - kb) : \\ j, k \in N;\ l = 1, \cdots, L\} \tag{2.1}$$

if there exist positive constant A and B, with $0 < A \leq B < \infty$, so that for $\forall f \in L^2(R)$, we have

$$A\|f\|_2^2 \leq \sum_{l=1}^{L} \sum_{j,k=1}^{\infty} |\langle f,\ \psi_{l,j,k}(x)\rangle|^2 \leq B\|f\|_2^2,$$

where A and B are called *upper and lower multi-wavelet frame bounds* respectively.

In the paper, we also need the following:

Lemma[3] *Let X and Y be Banach space, λ_1, $\lambda_2 \in [0, 1)$, and S, T: $X \to Y$ are linear operators so that for $\forall x \in X$, there is $\|S(x) - T(x)\| \leq \lambda_1\|S(x)\| + \lambda_2\|T(x)\|$, then there exists*

$$\mathrm{codim}_Y \overline{S(x)} = \mathrm{codim}_Y \overline{T(x)}.$$

3. The Stability of Multi-wavelet Frames

Theorem 3.1. *Let F_a defined in (2.1) be the multi-wavelet frames generated by*

$$\Psi_L(x) = \{\psi_1(x), \cdots, \psi_L(x)\},$$

let $T \in L(L^2(R),\ L^2(R))$ be surjection so that $T\psi_L(x) = \tilde{\psi}_L(x)$, $l = 1, \cdots, L$, then

$$\tilde{F}_a = \{\tilde{\psi}_{l,j,k}(x) = a^{\frac{j}{2}}\tilde{\psi}_l(a^j x - kb) : \\ j,\ k \in N,\ l = 1, \cdots, L\}$$

are the multi-wavelet frames of $L^2(R)$.

Proof. Since $F_a = \{\psi_{l,j,k}(x)\}_{l=1,\cdots,L;\, j,\, k\in N}$ is wavelet frames of $L^2(R)$, then for $\forall\, f \in L^2(R)$, there is

$$A\|f\|_2^2 \leq \sum_{l=1}^{L} \sum_{j,k=1}^{\infty} |\langle f,\ \psi_{l,j,k}(x)\rangle|^2 \leq B\|f\|_2^2.$$

By virtue of $T\psi_l(x) = \tilde{\psi}_l(x)$, $l = 1, \cdots, L$, there is

$$T\psi_{l,j,k}(x) = \tilde{\psi}_{l,j,k}(x), \quad j,\, k \in N;\ l = 1, \cdots, L.$$

In the following, we prove that for $\{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$, the inequality of frames holds true,

$$\begin{aligned}\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}\left|\left\langle g,\ \tilde{\psi}_{l,j,k}(x)\right\rangle\right|^2 &= \sum_{l=1}^{L}\sum_{j,k=1}^{\infty}|\langle g,\ T\psi_{l,j,k}(x)\rangle|^2\\ &= \sum_{l=1}^{L}\sum_{j,k=1}^{\infty}|\langle T^*g,\ \psi_{l,j,k}(x)\rangle|^2\\ &\le B\|T^*g\|_2^2 \le B\|T\|_2^2\|g\|_2^2\end{aligned}$$

and

$$\begin{aligned}\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}\left|\left\langle g,\ \tilde{\psi}_{l,j,k}(x)\right\rangle\right|^2 &= \sum_{l=1}^{L}\sum_{j,k=1}^{\infty}|\langle g,\ T\psi_{l,j,k}(x)\rangle|^2\\ &= \sum_{l=1}^{L}\sum_{j,k=1}^{\infty}|\langle T^*g,\ \psi_{l,j,k}(x)\rangle|^2\\ &\ge A\|T^*g\|_2^2 \ge A\left\|(T^*)^{-1}\ |_{T^*(L^2(R))}\right\|_2^2\ \|g\|_2^2\end{aligned}$$

then $\tilde{F}_a = \{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$ is the wavelet frames of $L^2(R)$. $\square$

Theorem 3.2. *Let the collection of functions $F_a = \{\psi_{l,j,k}(x)\}_{j,\,k\in N; j=1,\cdots,L}$ defined in* (2.1) *be the multi-wavelet frames which are generated by $\Psi_L(x)$ and the frame bounds are A and B, let $M \ge 0$, $\lambda \ge 0$, $0 \le \beta \le 1$ and $(1-\lambda)\sqrt{A} > \sqrt{M}$, and let $\{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$ be formed by $\tilde{\Psi}_L(x) = \{\tilde{\psi}_1(x), \cdots, \tilde{\psi}_L(x)\}$ by dilation and translation with the same dilation and translation parameters as $\{\psi_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots L}$ by $\Psi_L(x)$ in* (2.1)*, and $\{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$ satisfies that for $l = 1, \cdots, L$; $j,\ k = 1, 2, \cdots$, and for every $C_{l,1,1}, \cdots, C_{l,j,k}$, there is*

$$\begin{aligned}\left\|\sum_{l=1}^{L}\sum_{j,k=1}^{\infty} C_{l,j,k}(\psi_{l,j,k}(x) - \tilde{\psi}_{l,j,k}(x))\right\|_2 &\le \lambda\left\|\sum_{l=1}^{L}\sum_{j,k=1}^{\infty} C_{l,j,k}\psi_{l,j,k}(x)\right\|_2\\ &\quad + \beta\left\|\sum_{l=1}^{L}\sum_{j,k=1}^{\infty} C_{l,j,k}\tilde{\psi}_{l,j,k}(x)\right\|_2 \qquad (3.1)\\ &\quad + \sqrt{M}\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}|C_{l,j,k}|^2\Big)^{\frac{1}{2}}\end{aligned}$$

then $\{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$ are multi-wavelet frames of $L^2(R)$, and the frame bounds are $\dfrac{[(1-\lambda)\sqrt{A}-\sqrt{M}]^2}{(1+\beta)^2}$ and $\dfrac{[(1+\lambda)\sqrt{B}-\sqrt{M}]^2}{(1-\beta)^2}$.

Proof. For j, $k = 1, 2, \cdots$; $l = 1, \cdots, L$ and every $C_{l,1,1}$, $C_{l,1,2}, \cdots$, $C_{l,n,n}$, from (3.1) there is

$$\Big\|\sum_{l=1}^{L}\sum_{j,k=1}^{n} C_{l,j,k}\tilde{\psi}_{l,j,k}(x)\Big\|_2 \le \frac{1+\lambda}{1-\beta}\Big\|\sum_{l=1}^{L}\sum_{j,k=1}^{n} C_{l,j,k}\psi_{l,j,k}(x)\Big\|_2 + \frac{\sqrt{M}}{1-\beta}(\sum_{l=1}^{L}\sum_{j,k=1}^{n}|C_{l,j,k}|^2)^{\frac{1}{2}}$$

By virtue of the Hahn-Banach Theorem, there exist $x^* \in (L^2(R))^* = L^2(R)$, with $\|x^*\|_2 = 1$, so that

$$\begin{aligned}\Big\|\sum_{l=1}^{L}\sum_{j,k=1}^{n} C_{l,j,k}\psi_{l,j,k}(x)\Big\|_2 &= x^*\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n} C_{l,j,k}\psi_{l,j,k}(x)\Big)\\ &\le \Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n}|C_{l,j,k}|^2\Big)^{\frac{1}{2}}\\ &\quad\times\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n}|x^*(\psi_{l,j,k}(x))|^2\Big)^{\frac{1}{2}}\\ &\le \sqrt{B}\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n}|C_{l,j,k}|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

So, there is

$$\begin{aligned}\Big\|\sum_{l=1}^{L}\sum_{j,k=1}^{n} C_{l,j,k}\tilde{\psi}_{l,j,k}(x)\Big\|_2 &\le \frac{1}{1-\beta}\Big(\sqrt{B}(1+\lambda)+\sqrt{M}\Big)\\ &\quad\times\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n}|C_{l,j,k}|^2\Big)^{\frac{1}{2}}\end{aligned}$$

For every $y^* \in (L^2(R))^* = L^2(R)$, from the above formula, there is

$$\begin{aligned}\sum_{l=1}^{L}\sum_{j,k=1}^{n}|y^*\Big(\tilde{\psi}_{l,j,k}(x)\Big)|^2 &= y^*\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n}\frac{|y^*(\tilde{\psi}_{l,j,k}(x)|^2}{y^*(\tilde{\psi}_{l,j,k}(x))}\tilde{\psi}_{l,j,k}(x)\Big)\\ &\le \frac{\|y^*\|_2}{1-\beta}\big[\sqrt{B}(1+\lambda)+\sqrt{M}\big]\\ &\quad\times\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n}|y^*(\tilde{\psi}_{l,j,k}(x)|^2\Big)^{\frac{1}{2}}\end{aligned}$$

So, we have

$$\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{n}\Big|y^*(\tilde{\psi}_{l,j,k}(x))\Big|^2\Big)^{\frac{1}{2}} \leq \frac{\|y^*\|_2}{1-\beta}[\sqrt{B}(1+\lambda)+\sqrt{M}]$$

Let $n\to\infty$, then we have

$$\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}\Big|y^*(\tilde{\psi}_{l,j,k}(x))\Big|^2 \leq \|y\|_2^2\frac{[\sqrt{B}(1+\lambda)+\sqrt{M}]^2}{(1-\beta)^2}.$$

That is, for $\forall\, f\in L^2(R)$, there is

$$\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}\Big|\Big\langle f,\ \tilde{\psi}_{l,j,k}(x)\Big\rangle\Big|^2 \leq \|f\|_2^2\frac{[\sqrt{B}(1+\lambda)+\sqrt{M}]^2}{(1-\beta)^2}$$

In the following, we are to prove the lower frame bound of $\{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\,l=1,\cdots,L}$ is $\dfrac{[(1-\lambda)\sqrt{A}-\sqrt{M}]^2}{(1+\beta)^2}$.

Let $T: L^2(R)\to l_2$ be

$$T(x)=\{\langle x,\ \psi_{l,j,k}(x)\rangle\}_{j,\,k\in N;\,l=1,\cdots,L},$$

then there is

$$\sqrt{A}\|x\|_2^2\leq\|T(x)\|_2^2\leq\sqrt{B}\|x\|_2^2,\ \forall\, x\in L^2(R).$$

Let $V=T(L^2(R))$, then T^{-1} is linear homeomorphism and $\|T^{-1}\|\leq\sqrt{A^{-1}}$, and V is the closed subspace of l_2, so we can let $P:\ l_2\to V$ be orthogonal projection.

Let $S=T^{-1}P:\ l_2\to L^2(R)$, then there is

$$\|S\|\leq\|T^{-1}\|\cdot\|P\|\leq\sqrt{A^{-1}}.$$

Let $S^*: L^2(R)\to l_2$ be the conjugate operator of S, and let

$$S^*(x)=\{C_{l,j,k}(x)\}_{j,\,k\in N;l=1,\cdots,L},$$

then there is

$$\begin{aligned}\Big(\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}|C_{l,j,k}(x)|^2\Big)^{\frac{1}{2}} &\leq \|S\|\cdot\|x\|_2\\ &\leq\sqrt{A^{-1}}\|x\|_2\end{aligned}\tag{3.2}$$

For every $y\in L^2(R)$, there is

$$\begin{aligned}\Big\langle\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}C_{l,j,k}(x)\psi_{l,j,k}(x),\ y\Big\rangle &= \sum_{l=1}^{L}\sum_{j,k=1}^{\infty}C_{l,j,k}(x)\,\langle\psi_{l,j,k}(x),\ y\rangle\\ &=\langle S^*(x),\ T(y)\rangle=\langle x,\ ST(y)\rangle\\ &=\langle x,\ T^{-1}PT(y)\rangle=\langle x,\ y\rangle\end{aligned}$$

So we have

$$x = \sum_{l=1}^{L} \sum_{j,k=1}^{\infty} C_{l,j,k}(x) \psi_{l,j,k}(x).$$

Let $L(x) = \sum_{l=1}^{L} \sum_{j,k=1}^{\infty} C_{l,j,k}(x) \tilde{\psi}_{l,j,k}(x)$. From (3.1) and(3.2), the linear operator $L: L^2(R) \to L^2(R)$ which satisfies that for $\forall x \in L^2(R)$, there is

$$\|x - L(x)\|_2 \le (\lambda + \frac{\sqrt{M}}{\sqrt{A}})\|x\|_2 + \beta \|L(x)\|_2$$

and

$$\begin{aligned} \frac{(1-\lambda)\sqrt{A} - \sqrt{M}}{(1+\beta)\sqrt{A}} \|x\|_2 &\le \|L(x)\|_2 \\ &\le \frac{(1+\lambda)\sqrt{A} + \sqrt{M}}{(1-\beta)\sqrt{A}} \|x\|_2. \end{aligned}$$

So, L is one to one and bounded linear operator, and $L(L^2(R))$ is closed subspace. By virtue of the Lemma, we can conclude that L is surjection. Therefore, L is invertible bounded linear operator and

$$\|L^{-1}\| \le \frac{(1+\beta)\sqrt{A}}{[(1-\lambda)\sqrt{A} - \sqrt{M}]}$$

For every $x,\ y \in L^2(R)$, there is

$$\begin{aligned} |\langle x,\ L^*(y)\rangle| = |\langle L(x),\ y\rangle| &= \left| \sum_{l=1}^{L} \sum_{j,k=1}^{\infty} C_{l,j,k}(x) \left\langle \tilde{\psi}_{l,j,k}(x),\ y \right\rangle \right| \\ &\le \Big(\sum_{l=1}^{L} \sum_{j,k=1}^{\infty} |C_{l,j,k}(x)|^2 \Big)^{\frac{1}{2}} \\ &\quad \times \Big(\sum_{l=1}^{L} \sum_{j,k=1}^{\infty} \left| \left\langle y,\ \tilde{\psi}_{l,j,k}(x) \right\rangle \right|^2 \Big)^{\frac{1}{2}} \\ &\le \sqrt{A^{-1}} \|x\|_2 \Big(\sum_{l=1}^{L} \sum_{j,k=1}^{\infty} \left| \left\langle y,\ \tilde{\psi}_{l,j,k}(x) \right\rangle \right|^2 \Big)^{\frac{1}{2}} \end{aligned}$$

So, there is

$$\begin{aligned} \|L^*(y)\|_2 &= \sup_{\|x\| \le 1} |\langle x,\ L^*(y)\rangle| \\ &\le \sqrt{A^{-1}} \Big(\sum_{l=1}^{L} \sum_{j,k=1}^{\infty} |\langle y,\ \tilde{\psi}_{l,j,k}(x)\rangle|^2 \Big)^{\frac{1}{2}} \end{aligned}$$

So, there is

$$\begin{aligned}\|y\|_2 &= \|(L^*)^{-1}L^*(y)\|_2 \le \|L^{-1}\| \cdot \|L^*(y)\|_2 \\ &\le \frac{(1+\beta)\sqrt{A}}{(1-\lambda)\sqrt{A}-\sqrt{M}}\sqrt{A^{-1}} \\ &\quad \times \Big(\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}\Big|\langle y,\ \tilde{\psi}_{l,j,k}(x)\rangle\Big|^2\Big)^{\frac{1}{2}}\end{aligned}$$

That is, for $\forall\, f \in L^2(R)$, there is,

$$\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}\Big|\langle f,\ \tilde{\psi}_{l,j,k}(x)\rangle\Big|^2 \quad \ge \|f\|_2^2\Big(\frac{(1-\lambda)\sqrt{A}-\sqrt{M}}{1+\beta}\Big)^2$$

□

As the result of Theorem 2*, we have the following:*

Theorem 3.3. *Let* $\{\psi_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$ *be the multi-wavelet frames generated by* $\Psi_L(x)$ *in* (2.1)*, the upper and lower bounds are* A *and* B*, and let*

$$\{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L} \subseteq L^2(R)$$

be formed by $\tilde{\Psi}_L(x) = \{\tilde{\psi}_1(x), \cdots, \tilde{\psi}_L(x)\}$ *by dilation and translation with the same dilation and translation parameters as* $\{\psi_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$ *by* $\Psi_L(x)$ *in* (2.1)*. If there exist* $R < A$*, so that for* $\forall\, f \in L^2(R)$*, we have*

$$\sum_{l=1}^{L}\sum_{j,k=1}^{\infty}\Big|\Big\langle f,\ \psi_{l,j,k}(x)-\tilde{\psi}_{l,j,k}(x)\Big\rangle\Big|^2 \le R\|f\|_2^2,$$

then $\{\tilde{\psi}_{l,j,k}(x)\}_{j,\,k\in N;\, l=1,\cdots,L}$ *are multi-wavelet frames of* $L^2(R)$*, and the upper and lower frame bounds are* $(\sqrt{A}-\sqrt{M})^2$ *and* $(\sqrt{B}+\sqrt{M})^2$.

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Biorthogonal Wavelets Associated with Two-Dimensional Interpolatory Function

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Abstract. To construct biorthogonal wavelets from two-dimensional interpolatory function, a large amount of computation is involved in traditional method. In this paper, a method is developed for constructing the biorthogonal wavelets. Masks of the biorthogonal wavelets are given explicitly. Neither the Gram-Schmidt processing nor the inverse of a nonsingular polynomial matrix is needed.

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1. Introduction

During the past few years, the construction of interpolatory scaling function has become of increasing interest (see, e.g. [1, 2, 3, 4]). However, there still does not exist a simple method to construct biorthogonal wavelets from the associated two-dimensional interpolatory function. In this paper, a method is presented for constructing biorthogonal wavelets from two-dimensional interpolatory function. Biorthogonal wavelet masks can be constructed explicitly.

Let $\phi(\mathbf{x}) \in L^2(R^2)$, which satisfies the following refinable equation

$$\phi(\mathbf{x}) = \sum_{\alpha \in Z^2} p_\alpha \phi(2\mathbf{x} - \alpha), \tag{1.1}$$

and let

$$V_0 = \mathrm{Span}_{L^2(R^2)}\{\phi(\mathbf{x} - \alpha),\ \alpha \in Z^2\},\ V_k = \{f(2^k\mathbf{x} - \alpha),\ f \in V_0,\ \alpha \in Z^2\}.$$

If $\hat{\phi}(0) \neq 0$, it was shown in [5] (see also [6]) that $\{V_k\}_{k \in Z}$, the sequence of subspace of $L^2(R^2)$, satisfies

$$\overline{\bigcup_{k \in Z} V_k} = L^2(R^2), \qquad \bigcap_{k \in Z} V_k = \{0\}.$$

If $\phi(\mathbf{x})$ and its shifts form a Riesz basis of V_0, the sequence of the subspaces $\{V_k\}_{k\in Z}$ forms a multiresolution analysis (MRA) of $L^2(R^2)$. $\phi(\mathbf{x})$ is called a scaling function.

If $\phi(\mathbf{x})$ is a continuous two-dimensional scaling function, which satisfies

$$\phi(\alpha) = \delta_{0,\alpha}, \ \ \alpha \in Z^2, \tag{1.2}$$

we say that $\phi(\mathbf{x})$ is an interpolatory scaling function.

In practical image processing, images are often first represented by sampling space V_k. When the pixel values of an image f are given, an image is normally (or easily) represented by

$$f_k = 2^k \sum_{\alpha\in Z^2} f(\frac{\alpha}{2^k})\phi(2^k\mathbf{x} - \alpha)$$

for a certain dilation level k. However, to apply the decomposition and reconstruction algorithm, one should use the function [3]

$$\sum_{\alpha\in Z^2} < f_k, 2^k\tilde{\phi}(2^k\mathbf{x} - \alpha) > 2^k\phi(2^k - \alpha).$$

This function is not the function f_k, unless the refinable function ϕ satisfies the condition (1.2). Hence, by using the sampling space generated by interpolatory refinable function, one simplifies (or reduces the errors of) the first step of the decomposition and reconstruction algorithm (see [3]). During the past few years, some excellent results on the construction of interpolatory functions have been published. For any positive integers N and $\tilde{N}$ with $N \leq \tilde{N}$, J. Kovačević and Sweldens [7] constructed an interpolatory mask satisfying sum rules of order $\tilde{N}$ and its dual mask satisfying sum rules of order N. In [2], H. Ji *et al* proposed a convolution method to construct refinable functions of arbitrary regularity which are dual to an interpolatory scaling function. For an interpolatory mask, B. Han [1] has provided an CBC algorithm to construct the dual masks which satisfy sum rules of any given order. However, up to now, there is still no simple method to construct the two-dimensional biorthogonal wavelets from the interpolatory refinable function. The method provided by H. Ji *et al.* [2] needs not only the Gram-Schmidt processing but also the computation of the inverse of a Laurent matrix. In this paper, we provide a method for constructing the two-dimensional biorthogonal wavelets from the interpolatory refinable function. The wavelet masks are given explicitly.

The two-dimensional biorthogonal wavelet system is introduced in Section 2. In Section 3, formulas are provided for constructing biorthogonal wavelet masks from two-dimensional interpolatory function. Example is also given to demonstrate this method. Finally, the conclusion is given in Section 4.

2. Two-Dimensional Biorthogonal Wavelet System

We call $\Delta = \{\alpha : p_\alpha \neq 0\}$ the support of $\{p_\alpha\}_{\alpha\in Z^2}$. For $\omega = (\omega_1, \omega_2) \in R^2$, $\alpha = (\alpha_1, \alpha_2) \in Z^2$, we denote $z_1 = e^{-i\omega_1}$, $z_2 = e^{-i\omega_2}$, $\mathbf{z}^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$. Let

$$P(\mathbf{z}) = P(z_1, z_2) = \frac{1}{4}\sum_{\alpha\in Z^2} p_\alpha \mathbf{z}^\alpha = \frac{1}{4}\sum_{\alpha\in Z^2} p_\alpha z_1^{\alpha_1} z_2^{\alpha_2}.$$

For scaling function $\phi(\mathbf{x})$, if there exists $\tilde{\phi}(\mathbf{x})$ such that

$$\tilde{\phi}(\mathbf{x}) = \sum_{\alpha\in Z^2} \tilde{p}_\alpha \tilde{\phi}(2\mathbf{x}-\alpha), \text{ and } <\phi(\mathbf{x}), \tilde{\phi}(\mathbf{x}-\alpha>=\delta_{\alpha,0},$$

then we call $\tilde{\phi}(\mathbf{x})$ the dual of $\phi(\mathbf{x})$. We denote $\tilde{P}(\mathbf{z}) = \sum_{\alpha\in Z^2} \tilde{p}_\alpha \mathbf{z}^\alpha$.

Suppose that $\psi^j(\mathbf{x})$, $\tilde{\psi}^j(\mathbf{x})$ $(j = 1, 2, 3)$ are biorthogonal wavelets associated with $\phi(\mathbf{x}), \tilde{\phi}(\mathbf{x})$, which satisfy

$$\psi^j(\mathbf{x}) = \sum_{\alpha\in Z^2} q_\alpha^j \phi(2\mathbf{x}-\alpha), \ \tilde{\psi}^j(\mathbf{x}) = \sum_{\alpha\in Z^2} \tilde{q}_\alpha^j \tilde{\phi}(2\mathbf{x}-\alpha),$$

then

$$\begin{aligned} &<\psi^j(\mathbf{x}), \tilde{\phi}(\mathbf{x}-\alpha)>=<\phi(\mathbf{x}), \tilde{\psi}^j(\mathbf{x}-\alpha)>=0, \\ &<\psi^{j_1}(\mathbf{x}), \tilde{\psi}^{j_2}(\mathbf{x}-\alpha)>=\delta_{\alpha,0}\delta_{j_1,j_2}, \end{aligned}$$

where $j, j_1, j_2 = 1, 2, 3$. We call $Q^j(\mathbf{z}) = \sum_{\alpha\in Z^2} q_\alpha^j \mathbf{z}^\alpha$ and $\tilde{Q}^j(\mathbf{z}) = \sum_{\alpha\in Z^2} \tilde{q}_\alpha^j \mathbf{z}^\alpha$ the wavelet masks. For convenience, we denote

$$m(\omega) = P(\mathbf{z}), \ \tilde{m}(\omega) = \tilde{P}(\mathbf{z}), \ m^j(\omega) = Q^j(\mathbf{z}), \ \tilde{m}^j(\omega) = \tilde{Q}^j(\mathbf{z}).$$

Let $\mu_0 = (0, 0), \mu_1 = (1, 0), \mu_2 = (0, 1), \mu_3 = (1, 1)$. By [4], we know that

$$\sum_{k=0}^{3} m(\omega + \mu_k\pi)\overline{\tilde{m}(\omega + \mu_k\pi)} = 1, \tag{2.1}$$

$$\sum_{k=0}^{3} m(\omega + \mu_k\pi)\overline{\tilde{m}^j(\omega + \mu_k\pi)} = 0, \tag{2.2}$$

$$\sum_{k=0}^{3} m^j(\omega + \mu_k\pi)\overline{\tilde{m}(\omega + \mu_k\pi)} = 0, \tag{2.3}$$

$$\sum_{k=0}^{3} m^{j_1}(\omega + \mu_k\pi)\overline{\tilde{m}^{j_2}(\omega + \mu_k\pi)} = \delta_{j_1,j_2}, \ j, j_1, j_2 = 1, 2, 3. \tag{2.4}$$

If (2.2)$\sim$(2.4) are satisfied, we call $\{q_\alpha^j\}_{\alpha\in Z^2}$, $\{\tilde{q}_\alpha^j\}_{\alpha\in Z^2}(j = 1, 2, 3)$ the biorthogonal wavelet filters associated with $\{p_\alpha\}_{\alpha\in Z^2}$, $\{\tilde{p}_\alpha\}_{\alpha\in Z^2}$. If (2.1)$\sim$(2.4) are satisfied, we say that $\{p_\alpha\}_{\alpha\in Z^2}$,$\{\tilde{p}_\alpha\}_{\alpha\in Z^2}$, $\{q_\alpha^j\}_{\alpha\in Z^2}$, $\{\tilde{p}_\alpha^j\}_{\alpha\in Z^2}$ form a two-dimensional biorthogonal wavelet system [8].

3. Construction of Biorthogonal Wavelets

We adopt the following notations

$$E_{0k}=\frac{1}{2}\sum_{\alpha\in Z^2}p_{2\alpha+\mu_k}\mathbf{z}^{2\alpha},\ E_{jk}=\frac{1}{2}\sum_{\alpha\in Z^2}q^j_{2\alpha+\mu_k}\mathbf{z}^{2\alpha},$$
$$\tilde{E}_{0k}=\frac{1}{2}\sum_{\alpha\in Z^2}\tilde{p}_{2\alpha+\mu_k}\mathbf{z}^{2\alpha},\ \tilde{E}_{jk}=\frac{1}{2}\sum_{\alpha\in Z^2}\tilde{q}^j_{2\alpha+\mu_k}\mathbf{z}^{2\alpha},\ R(z_1,z_2)=(1,z_1,z_2,z_1z_2)^T,$$

where $k=0,1,2,3,\ j=1,2,3$. It follows that

$$P(\mathbf{z})=\frac{1}{2}(E_{00},E_{01},E_{02},E_{03})R(z_1,z_2), \tag{3.1}$$
$$\tilde{P}(\mathbf{z})=\frac{1}{2}(\tilde{E}_{00},\tilde{E}_{01},\tilde{E}_{02},E_{03})R(z_1,z_2), \tag{3.2}$$
$$Q^j(\mathbf{z})=\frac{1}{2}(E_{j0},E_{j1},E_{j2},E_{j3})R(z_1,z_2), \tag{3.3}$$
$$\tilde{Q}^j(\mathbf{z})=\frac{1}{2}(\tilde{E}_{j0},\tilde{E}_{j1},\tilde{E}_{j2},E_{03})R(z_1,z_2), \tag{3.4}$$

where $j=1,2,3,\ k=0,1,2,3$. We call $E_{0k},\tilde{E}_{0k},E_{jk},\tilde{E}_{jk}$ the polyphase factors of $P(\mathbf{z})$, $\tilde{P}(\mathbf{z})$, $Q^j(\mathbf{z})$, and $\tilde{Q}^j(\mathbf{z})$ respectively. Hence, the necessary condition for $\{p_\alpha\}_{\alpha\in Z^2}$, $\{\tilde{p}_\alpha\}_{\alpha\in Z^2}$, $\{q^j_\alpha\}_{\alpha\in Z^2}$, $\{\tilde{p}^j_\alpha\}_{\alpha\in Z^2}$ to form a biorthogonal wavelet system is that the following equation holds

$$E(\mathbf{z})\tilde{E}(\mathbf{z})^*=I_4, \tag{3.5}$$

where

$$E(\mathbf{z})=\begin{pmatrix} E_{00} & E_{01} & E_{02} & E_{03}\\ E_{10} & E_{11} & E_{12} & E_{13}\\ E_{20} & E_{21} & E_{22} & E_{23}\\ E_{30} & E_{31} & E_{32} & E_{33}\end{pmatrix},\ \tilde{E}(\mathbf{z})=\begin{pmatrix} \tilde{E}_{00} & \tilde{E}_{01} & \tilde{E}_{02} & \tilde{E}_{03}\\ \tilde{E}_{10} & \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13}\\ \tilde{E}_{20} & \tilde{E}_{21} & \tilde{E}_{22} & \tilde{E}_{23}\\ \tilde{E}_{30} & \tilde{E}_{31} & \tilde{E}_{32} & \tilde{E}_{33}\end{pmatrix}.$$

The matrixes $E(\mathbf{z})$, $\tilde{E}(\mathbf{z})$ are called the polyphase matrixes of biorthogonal wavelet system. Therefore, in order to construct the biorthogonal wavelets associated with $\phi(\mathbf{x})$ and $\tilde{\phi}(\mathbf{x})$, we only need to extend $E(\mathbf{z})$, $\tilde{E}(\mathbf{z})$ by their first rows so that Eq.(3.5) is satisfied.

Now we provide the formulas for constructing biorthogonal wavelet masks from two-dimensional interpolatory function.

Theorem 3.1. Suppose that $\phi(\mathbf{x})$ is an interpolatory scaling function, $\tilde{\phi}(\mathbf{x})$ is a dual of $\phi(\mathbf{x})$, E_{0k}, $\tilde{E}_{0k}$ $(k=0,1,2,3)$ are polyphase factors of $P(\mathbf{z})$, and $\tilde{P}(\mathbf{z})$. Then the biorthogonal wavelet masks $Q^j(\mathbf{z})$ and $\tilde{Q}^j(\mathbf{z})$ can be given as follows

$$Q^j(\mathbf{z}) = \frac{1}{2}(e_j^T - \overline{\tilde{E}_{0j}}(E_{00}\ E_{01}\ E_{02}\ E_{03}))R(z_1, z_2), \tag{3.6}$$

$$\tilde{Q}^j(\mathbf{z}) = \frac{1}{2}(e_j^T - (\overline{2E_{0j}}, 0, 0, 0))R(z_1, z_2), \tag{3.7}$$

where $j = 1, 2, 3$, and $e_1 = (0, 1, 0, 0)^T, e_2 = (0, 0, 1, 0)^T, e_3 = (0, 0, 0, 1)^T$.

Proof. By [4], we only need to prove that $Q^j(\mathbf{z})$ and $\tilde{Q}^j(\mathbf{z})$, which are given in (3.6) and (3.7), satisfy the conditions (2.2)$\sim$(2.4).

For $j = 1, 2, 3$, let

$$(E_{j0}, E_{j1}, E_{j2}, E_{j3}) = e_j^T - \overline{\tilde{E}_{0j}}(E_{00}\ E_{01}\ E_{02}\ E_{03}), \tag{3.8}$$

$$(\tilde{E}_{j0}, \tilde{E}_{j1}, \tilde{E}_{j2}, \tilde{E}_{j3}) = e_j^T - (\overline{2E_{0j}}, 0, 0, 0). \tag{3.9}$$

We note that $R(z_1, z_2)^T \overline{R(z_1, z_2)} = 4$. By (3.1)$\sim$(3.4), we only need to prove that $E_{jk}, \tilde{E}_{jk}$ $(k = 0, 1, 2, 3,\ j = 1, 2, 3)$, which are given in (3.8) and (3.9), satisfy the condition (3.5). Note that $\phi(\mathbf{x})$ is an interpolatory filter, by equation (1.2), we know that $p_{2\alpha} = \delta_{0,\alpha}$. It follows that $E_{00} = \frac{1}{2}$. Hence,

$$\begin{aligned} & (E_{j0}, E_{j1}, E_{j2}, E_{j3})(\tilde{E}_{j0}, \tilde{E}_{j1}, \tilde{E}_{j2}, \tilde{E}_{j3})^* \\ = & (e_j^T - \overline{\tilde{E}_{0j}}(E_{00}\ E_{01}\ E_{02}\ E_{03}))(e_j^T - (2E_{0j}, 0, 0, 0))^T \\ = & \delta_{jl} - \overline{\tilde{E}_{0j}} E_{0l} + 2E_{00}E_{0l}\overline{\tilde{E}_{0j}} = \delta_{jl}, \end{aligned}$$

where $j, l = 0, 1, 2, 3$. This completes the proof. □

Example. We consider the interpolatory filter and its dual filter in Example 6.5 of [1]. The mask of the interpolatory function is supported on $[-1, 1] \cap Z^2$ and given by

$$\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

The mask of its dual is supported on $[-4, 4] \cup Z^2$ and given by

$$\begin{bmatrix} 0 & 0 & 0 & \frac{3}{128} & \frac{3}{64} & \frac{3}{128} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{64} & -\frac{3}{32} & -\frac{3}{64} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{16} & -\frac{1}{8} & -\frac{3}{8} & -\frac{1}{8} & \frac{1}{16} & 0 & 0 \\ \frac{3}{128} & -\frac{3}{64} & -\frac{1}{8} & \frac{11}{32} & \frac{51}{64} & \frac{11}{32} & -\frac{1}{8} & -\frac{3}{64} & \frac{3}{128} \\ \frac{3}{64} & -\frac{3}{32} & -\frac{3}{8} & \frac{51}{64} & \frac{33}{16} & \frac{51}{64} & -\frac{3}{8} & -\frac{3}{32} & \frac{3}{64} \\ \frac{3}{128} & -\frac{3}{64} & -\frac{1}{8} & \frac{11}{32} & \frac{51}{64} & \frac{11}{32} & -\frac{1}{8} & -\frac{3}{64} & \frac{3}{128} \\ 0 & 0 & -\frac{1}{16} & -\frac{1}{8} & -\frac{3}{8} & -\frac{1}{8} & \frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{64} & -\frac{3}{32} & -\frac{3}{64} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{128} & \frac{3}{64} & \frac{3}{128} & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 3.1, the biorthogonal wavelet filters can be given as follows:

- $\{q^1_\alpha\}_{\alpha\in Z^2}$, $\{q^2_\alpha\}_{\alpha\in Z^2}$, $\{q^3_\alpha\}_{\alpha\in Z^2}$, their support are in $[-5, 5]$, which can be given as follows

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{128} & \frac{3}{64} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{32} & \frac{3}{32} & \frac{3}{64} & 0 & 0 & 0 & 0 \\
-\frac{3}{256} & -\frac{3}{256} & \frac{1}{32} & \frac{1}{16} & -\frac{51}{256} & -\frac{45}{128} & \frac{7}{128} & \frac{1}{16} & -\frac{3}{512} & -\frac{3}{256} & 0 \\
-\frac{3}{256} & -\frac{3}{128} & \frac{13}{256} & \frac{1}{8} & -\frac{43}{128} & -\frac{51}{64} & -\frac{43}{128} & \frac{1}{8} & \frac{13}{256} & -\frac{3}{128} & -\frac{3}{256} \\
-\frac{3}{512} & -\frac{3}{128} & \frac{13}{512} & \frac{1}{8} & -\frac{43}{256} & \frac{13}{64} & -\frac{35}{256} & \frac{1}{8} & \frac{13}{512} & -\frac{3}{128} & -\frac{3}{512} \\
-\frac{3}{256} & -\frac{3}{128} & \frac{13}{256} & \frac{1}{8} & -\frac{43}{128} & -\frac{51}{64} & -\frac{43}{128} & \frac{1}{8} & \frac{13}{256} & -\frac{3}{128} & -\frac{3}{256} \\
0 & -\frac{3}{256} & -\frac{3}{512} & \frac{1}{16} & \frac{7}{128} & -\frac{45}{128} & -\frac{51}{256} & \frac{1}{16} & \frac{1}{32} & -\frac{3}{256} & -\frac{3}{512} \\
0 & 0 & 0 & 0 & \frac{3}{64} & \frac{3}{32} & \frac{3}{64} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{64} & \frac{3}{128} & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{3}{512} & -\frac{3}{256} & -\frac{3}{512} & -\frac{3}{256} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3}{256} & -\frac{3}{128} & -\frac{3}{128} & -\frac{3}{128} & -\frac{3}{256} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{32} & \frac{13}{256} & \frac{13}{512} & \frac{13}{256} & -\frac{3}{512} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} & 0 & 0 \\
0 & 0 & \frac{3}{128} & \frac{3}{64} & -\frac{51}{256} & -\frac{43}{128} & -\frac{43}{256} & -\frac{43}{128} & \frac{7}{128} & \frac{3}{64} & 0 \\
0 & 0 & \frac{3}{64} & \frac{3}{32} & -\frac{45}{128} & -\frac{51}{64} & \frac{13}{64} & -\frac{5}{64} & -\frac{45}{128} & \frac{3}{32} & \frac{3}{64} \\
0 & 0 & 0 & \frac{3}{64} & \frac{7}{128} & -\frac{43}{128} & -\frac{43}{256} & -\frac{43}{128} & -\frac{51}{256} & \frac{3}{64} & \frac{3}{128} \\
0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3}{512} & \frac{13}{256} & \frac{13}{512} & \frac{13}{256} & \frac{1}{32} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3}{256} & -\frac{3}{128} & -\frac{3}{128} & -\frac{3}{128} & -\frac{3}{256} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3}{256} & -\frac{3}{512} & -\frac{3}{256} & -\frac{3}{512} & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{256} & \frac{3}{128} & \frac{3}{256} & \frac{3}{128} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{128} & \frac{3}{64} & \frac{3}{64} & \frac{3}{64} & \frac{3}{128} & 0 & 0 \\
0 & 0 & \frac{3}{256} & \frac{3}{128} & -\frac{11}{128} & -\frac{19}{128} & -\frac{19}{256} & -\frac{19}{128} & \frac{3}{128} & \frac{3}{128} & 0 \\
0 & 0 & \frac{3}{128} & \frac{3}{64} & -\frac{19}{128} & -\frac{11}{32} & -\frac{11}{32} & -\frac{11}{32} & -\frac{19}{128} & \frac{3}{64} & \frac{3}{128} \\
0 & 0 & \frac{3}{256} & \frac{3}{64} & -\frac{19}{256} & -\frac{11}{32} & \frac{53}{64} & -\frac{11}{32} & -\frac{19}{256} & \frac{3}{64} & \frac{3}{256} \\
0 & 0 & \frac{3}{128} & \frac{3}{64} & -\frac{19}{128} & -\frac{11}{32} & -\frac{11}{32} & -\frac{11}{32} & -\frac{19}{128} & \frac{3}{64} & \frac{3}{128} \\
0 & 0 & 0 & \frac{3}{128} & \frac{3}{128} & -\frac{19}{128} & -\frac{19}{256} & -\frac{19}{128} & -\frac{11}{128} & \frac{3}{128} & \frac{3}{256} \\
0 & 0 & 0 & 0 & \frac{3}{128} & \frac{3}{64} & \frac{3}{64} & \frac{3}{64} & \frac{3}{128} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{128} & \frac{3}{256} & \frac{3}{128} & \frac{3}{256} & 0 & 0
\end{bmatrix}$$

- $\{\tilde{q}^1_\alpha\}_{\alpha\in Z^2}$, $\{\tilde{q}^1_\alpha\}_{\alpha\in Z^2}$, $\{\tilde{q}^1_\alpha\}_{\alpha\in Z^2}$, their support are all in $[0,3]^2$, which can be given as follows

$$\begin{bmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

Remark 3.2. In the construction of multi-dimensional wavelets, to improve the properties of the wavelets, the support of the scaling function may be enlarged. As a result, the nonzero coefficients in $\{p_\alpha\}_{\alpha\in Z^2}$ and $\{\tilde{p}_\alpha\}_{\alpha\in Z^2}$ will increase drastically. If the biorthogonal wavelets are constructed by the method given in [2], not only the Gram-Schmidt processing but also the inverse of a nonsingular polynomial matrix is needed. In our method, the biorthogonal wavelet masks are given explicitly (see the formulas (3.6) and (3.7)). Hence, the procedure for constructing two-dimensional biorthogonal wavelets from an interpolatory function is similar to that of one-dimensional biorthogonal wavelets. When compared with [2], our method results in significant amount of computational saving.

Remark 3.3. In this paper, we only consider the construction of two-dimensional biorthogonal wavelets associated with the dilation matrix $2I$. In fact, Theorem 3.1 can be generalized to the case of multidimensional and arbitrary dilation matrix.

4. Conclusion

For a pair of two-dimensional dual refinable functions, when one of them is interpolatory, we provide formulas for constructing the associated biorthogonal wavelet masks. Neither the Gram-Schmidt processing nor the computation of the inverse of a Laurent matrix is needed.

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Parameterization of Orthogonal Filter Bank with Linear Phase

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Abstract. For the M-channel FIR orthogonal filter bank with linear phase, a complete parameterization is obtained by applying the singular value decomposition of matrices related to the corresponding polyphase matrix. In the obtained parameterization forms, the number of the required parameters is reduced to $(N+2)\binom{\frac{M}{2}}{2}$.

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1. Introduction

In some applications, particularly in image processing, it is desirable that each individual filter in a paraunitary system is of linear phase. Paraunitary FIR filter banks can perform an orthogonal transformation to the data without phase distortion, and this symmetric property(linear phase) can be used for efficient implementation[1], so we only discuss the paraunitary FIR filter banks with linear phase in this paper.

For general paraunitary filter banks, Vaidyanathan et. al. proposed a complete and minimal structure[2][3], which shows that any parauintary system of McMillan degree N can be factorized into a product of N degree-one paraunitary building blocks and an unitary matrix. In practice, we care more about the filter length than the degree of the system. In this case, it is expected to have a parameterization form for paraunitary systems with given order[4].

In order to get the parameterization of a paraunitary system with linear phase and order N, by applying the theory of the balanced vectors, [5] obtained the form (3.9) which requires $2(N+2)\binom{M/2}{2}$ parameters; applying the singular

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value decomposition of the matrices, [4] obtained the form (9) which requires $2(N+1)\binom{M/2}{2}$ parameters, moreover, by using the method similar to the cosine-sine decomposition of a general unitary matrix, [4] further factorized this system into the form (47) which requires $(N+2)\binom{M/2}{2}$ parameters.

Motivated by [6], we propose a complete parameterization of a paraunitary system with order N, only by applying the singular value decomposition of the matrices related to the coefficient matrices of this system. Our aim is such that the required parameters representing this system is as small as possible by a more efficient method.

In addition, in M-band orthogonal wavelet systems, there are $M-1$ wavelet filters, and they are not uniquely determined by the scaling filter, it is very difficult to construct these wavelet filters, the parameterization of the filter bank is an important method to construct these wavelet filters. And the complete parameterization of the filter bank often provides an efficient structure for optimal design and fast implementation too, thus the research on the complete parameterization of the paraunitary filter bank with linear phase is very important too.

2. Prepared Knowledge

A filter $H_0(z)$ is called an M-band orthogonal scaling filter if it satisfies the condition

$$\sum_{k=0}^{M-1} \left|H_0(e^{i(\frac{\omega+2k\pi}{M})})\right|^2 = 1. \tag{2.1}$$

Given a scaling filter, the associated scaling function $\psi_0(x)$ is the solution of the following two-scale refinement equation

$$\psi_0(x) = \sum_n h_0(n)\psi_0(Mx-n). \tag{2.2}$$

A sufficient condition for (2.2) to have a solution in $L^1(R)$ is that the scaling filter $H_0(z)$ satisfies the linear constraint

$$H_0(1) = 1, \tag{2.3}$$

and

$$H_0(e^{i\omega}) \neq 0, \quad \forall\, \omega \in [-\frac{\omega}{M}, \frac{\omega}{M}], \tag{2.4}$$

the associated scaling function generates an orthogonal basis.

Corresponding to the scaling filter $H_0(z)$, there are $M-1$ wavelet filters $H_l(z)$, and they satisfy

$$\sum_{k=0}^{M-1} H_p(e^{i(\frac{\omega+2k\pi}{M})})H_q^*(e^{i(\frac{\omega+2k\pi}{M})}) = \delta_{p-q}, \quad 0 \le p \le M-1,\ 1 \le q \le M-1, \tag{2.5}$$

the corresponding wavelet functions are defined by

$$\widehat{\psi}_l(\omega) = H_l(e^{i\omega/M})\,\widehat{\phi}(\omega/M),\ l = 1, 2, \cdots, M-1.$$

We call $\mathcal{H}(z) = [H_0(z), H_1(z), \cdots, H_{M-1}(z)]^T$ the filter bank of the M-band wavelet system. When $H_l(z)(0 \le l \le M-1)$ satisfies (2.1) and (2.5), we say that $\mathcal{H}(z)$ is orthogonal. Here, we only discuss real FIR causal filter bank and M is even.

If every filter of the filter bank $\mathcal{H}(z)$ is either symmetric or antisymmetric about the same center of symmetry $(M(N+1)-1)/2$, i.e.,

$$h_k(M(N+1)-1-n) = s_k h_k(n), \tag{2.6}$$

where $s_k = 1$ or -1, $0 \le n \le M(N+1)-1$, $0 \le k \le M-1$, then we call $\mathcal{H}(z)$ to have linear phase.

The filter bank $\mathcal{H}(z)$ can also be represented by its polyphase matrix $P(z)$ as follows

$$\mathcal{H}(z) = \frac{1}{\sqrt{M}} P(z^M) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}, \tag{2.7}$$

where

$$P(z) = \begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) & \cdots & H_{0,M-1}(z) \\ \cdots & \cdots & \cdots & \cdots \\ H_{M-1,0}(z) & H_{M-1,1}(z) & \cdots & H_{M-1,1}(z) \end{bmatrix},$$

$$H_k(z) = \frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} H_{k,l}(z^m) z^{-k}, \quad H_{k,l}(z) = \sqrt{M} \sum_{n=0}^{N} h_k(Mn+l) z^{-n}.$$

Then the filter bank $\mathcal{H}(z)$ is orthogonal if and only if the corresponding polyphase matrix $P(z)$ is paraunitary[6], i.e., $P(z)$ satisfies

$$P(z^{-1})^T P(z) = I_M,$$

meanwhile, $P(z)$ is said to form a paraunitary system.

Since the polyphase representation is useful not only for theoretical study of filter banks but also for their efficient implementation, it is reasonable to require the symmetries of filters to conform to the polyphase structure. In other words, in order to obtain the parameterization of paraunitary filter banks with linear phase, we first need to obtain the characterization of its polyphase matrix which reflects the linear phase property of the individual filters. By the simple computation, we can obtain the following lemma[5]

Lemma 2.1. *Let $P(z)$ be the polyphase matrix of the filter bank $\mathcal{H}(z)$, then the filter $h_l(k)$($0 \le l \le M-1$) has linear phase if and only if $P(z)$ satisfies the following condition*

$$P(z) = z^{-N} D P(z^{-1}) J_M, \tag{2.8}$$

where $D = \mathrm{diag}(s_0, s_1, \cdots, s_{M-1})$, J_M is $M \times M$ antidiagonal matrix, N is the order of $P(z)$, i.e., the highest power of z^{-1} in $P(z)$.

We can obtain the parameterization of another paraunitary matrix $E_N(z)$ that is a transform of $P(z)$ by applying the singular value decomposition of matrices, so does the one of $P(z)$ by the relation between $P(z)$ and $E_N(z)$.

3. Main Results

In this section, we shall give the parameterization of the orthogonal filter bank $\mathcal{H}(z)$ with linear phase, by the method of the singular value decomposition of matrices, here, the corresponding polyphase matrix $P(z)$ satisfies (2.8).

Let the unitary matrix

$$R = \frac{\sqrt{2}}{2} \begin{bmatrix} I_{M/2} & J_{M/2} \\ J_{M/2} & -I_{M/2} \end{bmatrix},$$

then

$$R^T J_M R = \mathrm{diag}(I_{M/2}, -I_{M/2}). \tag{3.1}$$

From [5], we know that there are $M/2$ symmetric, and $M/2$ antisymmetric filters in the filter bank $\mathcal{H}(z)$ when M is even, thus the half of the diagonal elements of D are 1, and the others are -1, so there exists a $M \times M$ permutation matrix Q such that

$$QDQ^T = \mathrm{diag}(I_{M/2}, -I_{M/2}). \tag{3.2}$$

Take

$$E_N(z) = QP(z)R, \tag{3.3}$$

by (2.8), (3.1) and (3.2), we have

$$E_N(z) = z^{-N}\mathrm{diag}(I_{M/2}, -I_{M/2})E_N(z^{-1})\,\mathrm{diag}(I_{M/2}, -I_{M/2}), \tag{3.4}$$

it follows that $E_N(z)$ is paraunitary by the paraunitarity of $P(z)$ and unitarity of R and Q.

Theorem 3.1. *A causal FIR matrix $E_N(z)$ is paraunitary and satisfies* (3.4) *if and only if $E_N(z)$ can be parameterized as*

$$E_N(z) = V_N(z)V_{N-1}(z)\cdots V_1(z)\,\mathrm{diag}(w, u), \tag{3.5}$$

where $V_i(z)(1 \le i \le N)$ has the following form

$$V_i(z) = \frac{1}{2} \begin{bmatrix} I_{M/2} & -v_i \\ -v_i^T & I_{M/2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} I_{M/2} & v_i \\ v_i^T & I_{M/2} \end{bmatrix} z^{-1}, \tag{3.6}$$

v_i, w and u are $\frac{M}{2} \times \frac{M}{2}$ orthogonal matrices.

Proof. Assume that

$$E_N(z) = e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots + e_N z^{-N}$$

be paraunitary and satisfy (3.4). It follows that

$$e_N = \mathrm{diag}(I_{M/2}, -I_{M/2})\, e_0\, \mathrm{diag}(I_{M/2}, -I_{M/2}). \tag{3.7}$$

Since $E_N(z)$ is paraunitary, then

$$e_N^T e_0 = 0. \tag{3.8}$$

By (3.7) and (3.8), we obtain that

$$e_0^T \,\mathrm{diag}(I_{M/2}, -I_{M/2})\, e_0 = 0. \tag{3.9}$$

Partitioning e_0 as $e_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, where both B_1 and B_2 are $\frac{M}{2} \times M$ matrices, thereby by (3.9), we have

$$e_0^T \,\mathrm{diag}(I_{M/2}, -I_{M/2})\, e_0 = \left[B_1^T, B_2^T\right] \,\mathrm{diag}(I_{M/2}, -I_{M/2}) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0,$$

furthermore,

$$B_1^T B_1 = B_2^T B_2. \tag{3.10}$$

Therefore, both B_1 and B_2 have the same singular values. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be their nonzero singular values, then $\sigma_i (1 \le i \le n)$ is positive. Furthermore, there exists an $M \times M$ orthogonal matrix G such that

$$B_1^T B_1 = B_2^T B_2 = G \,\mathrm{diag}(\Lambda_n^2, 0)\, G^T, \tag{3.11}$$

where $\Lambda_n = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$.

From (3.11), it follows that

$$\mathrm{rank} B_1 = \mathrm{rank} B_1^T B_1 = \mathrm{rank} B_2 = \mathrm{rank} B_1^T B_1 = n \le \frac{M}{2},$$

and there are two $\frac{M}{2} \times \frac{M}{2}$ orthogonal matrices G_1, G_2 such that

$$B_1 B_1^T = G_1 \,\mathrm{diag}(\Lambda_n^2, 0)\, G_1^T; \quad B_2 B_2^T = G_2 \,\mathrm{diag}(\Lambda_n^2, 0)\, G_2^T. \tag{3.12}$$

Thereby by means of (3.11), (3.12) and the theory of singular value decomposition(see [8]: 144-147), we can derive that the singular value decompositions of B_1, B_2 are

$$B_1 = G_1 \,\mathrm{diag}(\Lambda_n, 0)\, G^T; \quad B_2 = G_2 \,\mathrm{diag}(\Lambda_n, 0)\, G^T, \tag{3.13}$$

then

$$B_1 = (G_1 G_2^T)\, G_2 \,\mathrm{diag}(\Lambda_n, 0)\, G^T = -v_N B_2, \tag{3.14}$$

where $v_N = -G_1 G_2^T$.

According to the orthogonal properties of G_1 and G_2, we obtain

$$v_N^T v_N = (-G_1 G_2^T)^T (-G_1 G_2^T) = I_{M/2},$$

namely, v_N is orthogonal and satisfies

$$[I_{M/2}, v_N] e_0 = [I_{M/2}, v_N] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = [I_{M/2}, v_N] \begin{bmatrix} -v_N B_2 \\ B_2 \end{bmatrix} = 0. \tag{3.15}$$

Moreover,

$$\left[v_N^T, I_{M/2}\right] e_0 = \left[v_N^T, I_{M/2}\right] \begin{bmatrix} -v_N B_2 \\ B_2 \end{bmatrix} = -v_N^T v_N B_2 + B_2 = 0,$$

so

$$\begin{bmatrix} I_{M/2} & v_N \\ v_N^T & I_{M/2} \end{bmatrix} e_0 = 0. \tag{3.16}$$

By (3.7) and (3.16), we have

$$\begin{aligned} &\begin{bmatrix} I_{M/2} & -v_N \\ -v_N^T & I_{M/2} \end{bmatrix} e_N \\ &= \begin{bmatrix} I_{M/2} & -v_N \\ -v_N^T & I_{M/2} \end{bmatrix} \operatorname{diag}(I_{M/2}, -I_{M/2})\, e_0 \operatorname{diag}(I_{M/2}, -I_{M/2}) \\ &= \begin{bmatrix} I_{M/2} & v_N \\ -v_N^T & -I_{M/2} \end{bmatrix} e_0 \operatorname{diag}(I_{M/2}, -I_{M/2}) \\ &= \operatorname{diag}(I_{M/2}, -I_{M/2}) \begin{bmatrix} I_{M/2} & v_N \\ v_N^T & I_{M/2} \end{bmatrix} e_0 \operatorname{diag}(I_{M/2}, -I_{M/2}) \\ &= 0. \end{aligned} \tag{3.17}$$

Take the order-one matrix $V_N(z)$ as (3.6), by the direct computation, we have that $V_N(z)$ satisfies the following properties:

$$i)\quad V_N(z)\, V_N(z^{-1}) = I_M; \tag{3.18}$$

$$ii)\quad V_N(z^{-1}) \operatorname{diag}(I_{M/2}, -I_{M/2})\, V_N(z^{-1}) = z \operatorname{diag}(I_{M/2}, -I_{M/2}). \tag{3.19}$$

Take

$$E_N(z) = V_N(z) E_{N-1}(z). \tag{3.20}$$

Since both $E_N(z)$ and $V_N(z)$ are paraunitary, then $E_{N-1}(z)$ is *paraunitary.* It is clear that $E_{N-1}(z)$ has *order* $\mathbf{N-1}$ in virtue of (3.17).

From (3.6), (3.18) and (3.20), it follows that

$$\begin{aligned} E_{N-1}(z) &= V_N(z^{-1}) E_N(z) \\ &= \frac{1}{2} \begin{bmatrix} I_{M/2} & -v_N \\ -v_N^T & I_{M/2} \end{bmatrix} E_N(z) + \frac{1}{2} z \begin{bmatrix} I_{M/2} & v_N \\ v_N^T & I_{M/2} \end{bmatrix} E_N(z). \end{aligned} \tag{3.21}$$

The second term on the right-hand side of (3.21) is responsible for the non-causality, but we select v_N such that (3.16) holds, therefore $E_N(z)$ is *causal.*

Substitute (3.20) into (3.4), we have

$$z^{-N} \operatorname{diag}(I_{M/2}, -I_{M/2})\, V_N(z^{-1})\, E_{N-1}(z^{-1}) \operatorname{diag}(I_{M/2}, -I_{M/2}) = V_N(z)\, E_{N-1}(z),$$

namely,

$$\begin{aligned} &z^{-N} \big(V_N(z^{-1}) \operatorname{diag}(I_{M/2}, -I_{M/2}) V_N(z^{-1})\big)\, E_{N-1}(z^{-1}) \operatorname{diag}(I_{M/2}, -I_{M/2}) \\ &= E_{N-1}(z), \end{aligned}$$

by (3.19), we know

$$z^{-(N-1)} \operatorname{diag}(I_{M/2}, -I_{M/2})\, E_{N-1}(z^{-1}) \operatorname{diag}(I_{M/2}, -I_{M/2}) = E_{N-1}(z),$$

therefore $E_{N-1}(z)$ satisfies (3.4), i.e., the corresponding filter bank has *linear phase.*

In a word, $E_{N-1}(z)$ is paraunitary, and has linear phase. At the moment, there is a reduction in order by 1, thereby for a paraunitary matrix $E_N(z)$ satisfying (3.4), this process is guaranteed to terminate in N steps. Similar to the above process, there exist $\frac{M}{2} \times \frac{M}{2}$ orthogonal matrices $v_{N-1}, \cdots, v_1$ such that $V_{N-1}(z), \cdots, V_1(z)$ defined by (3.6) satisfy the properties (3.18) and (3.19). Then $E_N(z)$ can be written as

$$E_N(z) = V_N(z)E_{N-1}(z) = V_N(z)\,V_{N-1}(z)E_{N-2}(z) = V_N(z)\cdots V_1(z)E_0, \quad (3.22)$$

where

$$E_0^T E_0 = I_M, \quad E_0 = \mathrm{diag}(I_{M/2}, -I_{M/2})\,E_0\,\mathrm{diag}(I_{M/2}, -I_{M/2}),$$

it follows that E_0 has the form $E_0 = \mathrm{diag}(w, u)$, w, u are $\frac{M}{2} \times \frac{M}{2}$ orthogonal matrices. Substitute E_0 into (3.22), we have that the necessity holds.

For the sufficiency, by (3.18), (3.19) and the hypothesis, it follows that $E_N(z)$ satisfying (3.5) is causal, paraunitary and satisfies (3.4). □

By the theorem 3.1 and (3.3), we derive

Corollary 3.2. *Let $P(z)$ be the corresponding polyphase matrix of the filter bank $\mathcal{H}(z)$. Then $P(z)$ is paraunitary and satisfies* (2.8) *if and only if $P(z)$ can be parameterized as*

$$P(z) = Q^T V_N(z)V_{N-1}(z)\cdots V_1(z)\,\mathrm{diag}(w, u)R^T \quad (3.23)$$

Through (2.7) and the corollary 3.2, we derive the following theorem

Theorem 3.3. *When M is even, a causal FIR filter bank $\mathcal{H}(z)$ is orthogonal and satisfies* (2.6) *if and only if it can be parameterized as*

$$\mathcal{H}(z) = \frac{1}{\sqrt{M}} Q^T V_N(z)V_{N-1}(z)\cdots V_1(z)\mathrm{diag}(w, u)\cdot R^T \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}. \quad (3.24)$$

By the parameterization form (3.24) of the filter bank $\mathcal{H}(z)$, we can design many scaling filters with different properties, and let they satisfy the conditions (2.3) and (2.4), then we derive scaling functions and the corresponding wavelets.

4. Comment

For our results, the parameterization form (3.23) of $P(z)$ requires $(N+2)\binom{M/2}{2}$ parameters, which is less than that of (3.9) in [5] and (9) in [4], and is equal to that of (47) in [4]. But we only use singular values decomposition of matrices to get the parameterization of $P(z)$ with the same parameters, out of question, not only this method decrease computing complexity, but it is simpler and more effective as well.

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On Multivariate Wavelets with Trigonometric Vanishing Moments

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Abstract. Wavelets with trigonometric vanishing moments are studied for the first time. A practical construction algorithm of multivariate orthonormal wavelets with trigonometric vanishing moments is proposed. Based on such construction algorithm, a tight frame of $L^2(\mathbb{R}^d)$ can be obtained even at the worst case. An example of construction of bivariate orthonormal wavelets providing concrete trigonometric vanishing moments is presented.

Keywords. Non-tensor product, trigonometric vanishing moment, tight framework, vanishing moment, trigonometric polynomial.

1. Introduction

Wavelets have been a rapidly developed branch in mathematics, which become a powerful tool for many applications such as signal processing, function approximation, image processing and computational molecule biology. With the remarkable increase of processing problems, it is very desirable to design wavelets possessing various properties such as orthonormality or symmetry or compact support or vanishing moments.

The order of vanishing moments is one of the most important factors for success of wavelets in various applications such as image compression and singularity detection. In particular, vanishing moments are necessary for smoothness of wavelets (see [1]) and guarantee the approximation order (see [2]).

It is well known that trigonometric polynomials can also accomplish the same outstanding approximation behavior compared with algebra polynomials. In this paper, trigonometric vanishing moments are studied. Wavelets with trigonometric vanishing moments are orthogonal to trigonometric polynomials. Such wavelets not only inherit the excellence of wavelets with vanishing moments, but also keep the good features of exponential (trigonometric) function basis, such as periodicity and specific frequency information.

For multivariate non-tensor product compactly supported orthonormal wavelets providing trigonometric vanishing moments, a practical construction algorithm is found. In addition, a tight frame of $L^2(\mathbb{R}^d)$ can be obtained even at the worst case utilizing our proposed construction algorithm. An example of construction of bivariate orthonormal wavelets providing concrete trigonometric vanishing moments is presented.

2. Notations and Preliminaries

A compactly supported multivariate function $f(x), x \in \mathbb{R}^d$ is refinable with mask c if it satisfies the following refinement equation:

$$f(x) = \sum_{j \in \mathbb{Z}^d} c_j f(2x - j) \tag{1}$$

where $c = \{c_j\}_{\mathbb{Z}^d}$ is a finitely supported sequence on $\mathbb{Z}^d$. If c satisfies $\sum_{j \in \mathbb{Z}^d} c_j = 2^d$, there exists a unique compactly supported function $f(x)$ such that the refinement equation holds [3]. Define $C(w)$ as a mask symbol of c in the following:

$$C(w) = \sum_{j \in \mathbb{Z}^d} c_j e^{-iwj}, w \in \mathbb{R}^d.$$

By δ we denote the Dirac sequence on $\mathbb{Z}^d$ defined by $\delta_0 = 1$ and $\delta_k = 0$ for all $k \in \mathbb{Z}^d \backslash \{0\}$. $l(\mathbb{Z}^d)$ denote the linear space of all sequences defined on $\mathbb{Z}^d$. Let $l^2(\mathbb{Z}^d)$ be a space of all sequences $\{a_l\}_{l \in \mathbb{Z}^d}$ which satisfy $\sum_{l \in Z^2} |a_l|^2 < +\infty$. Let $\mathbf{E}$ denote the 2^d vertexes set of the d-dimension hypercube. $\mathbf{F} = \mathbf{E} \backslash \{0\}$. The elements of $\mathbf{E}$ beginning with zero are sorted in order. Define $\delta_{\nu,\mu} = 1,$ if $\nu = \mu;$ otherwise $\delta_{\nu,\mu} = 0$. Let $|\alpha| = \sum_{i=1}^{d} |\alpha_i|$ be the length of $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}^d$.

Define a shift operator $E^y, y \in \mathbb{R}^d$ as follows

$$E^y g(x) = g(x + y) \tag{2}$$

where g is a function on $\mathbb{R}^d$.

For any nonnegative integer N, denote T^d_{N-1} as the space of all d-dimension trigonometric polynomials of total degree less than N. Let $T^d_{-1} = 0$.

A compactly supported function $f \in L^1(\mathbb{R}^d)$ has N order vanishing moments if

$$\int_{\mathbb{R}^d} f(x)p(x)dx = 0, \qquad \forall p \in \Pi_{N-1} \tag{3}$$

where Π_{N-1} is the set of all polynomials of total degree less than N.

Compared with vanishing moments, the definition of N order trigonometric vanishing moments is proposed in the following.

Definition 2.1. *A compactly supported function $f \in L^1(\mathbb{R}^d)$ has N order trigonometric vanishing moments if*

$$\int_{\mathbb{R}^d} f(x)p(x)dx = 0, \qquad \forall p \in T^d_{N-1} \tag{4}$$

To avoid confusions, we emphasize that the wavelets with vanishing moments are orthogonal to polynomials.

3. Construction of Orthonormal Wavelets with Trigonometric Vanishing Moments

Let compactly supported functions ϕ and $\psi^\mu, \mu \in \mathbf{F}, \in L^2(\mathbb{R}^d)$ be scaling function and wavelet functions respectively, trigonometric polynomial functions $H(w) = \frac{1}{2^d}\sum\limits_{n\in\mathbb{Z}^d} h_n e^{-iwn}$ and $G^\mu(w) = \frac{1}{2^d}\sum\limits_{n\in\mathbb{Z}^d} g^\mu_n e^{-iwn}, \mu \in \mathbf{F}$, are the corresponding refinement filter symbols. So we have:

$$\hat{\phi}(w) = H(w/2)\hat{\phi}(w/2) \tag{5}$$

$$\hat{\psi}^\mu(w) = G^\mu(w/2)\hat{\phi}(w/2), \qquad \mu \in \mathbf{F} \tag{6}$$

For d-dimension orthonormal wavelets, the following necessary conditions are easily obtained.

$$H(0) = 1, \qquad \textstyle\sum_{\tau\in\mathbf{E}} |H(\pi\tau + w)|^2 = 1 \tag{7}$$

$$\begin{array}{l} \sum_{\tau\in\mathbf{E}} H(\pi\tau + w)\overline{G^\mu(\pi\tau + w)} = 0 \\ \sum_{\tau\in\mathbf{E}} G^\nu(\pi\tau + w)\overline{G^\mu(\pi\tau + w)} = \delta_{\nu,\mu}, \ \nu, \mu \in \mathbf{F} \end{array} \tag{8}$$

When the scaling filter symbol $H(w)$ satisfies the formula (7), we call that $H(w)$ is orthonormal. The key of construction of orthonormal wavelets is to solve $H(w)$ and $G^\mu, \mu \in \mathbf{F}$, satisfying (7) and (8).

If the wavelets provides trigonometric vanishing moments, we can easily deduce the following conclusions.

Theorem 3.1. *The wavelet functions $\psi^\mu, \mu \in \mathbf{F}$, have N order trigonometric vanishing moments if only and if* $\hat{\psi}^\mu(k) = 0, |k| < N, k \in \mathbb{Z}^d, \ \mu \in \mathbf{F}$.

Lemma 3.1. *For $\mu \in \mathbf{F}$, let trigonometric polynomial $G^\nu(w)$ be the corresponding refinement filter symbol of the wavelet function ψ^μ. If $G^\nu(w), \nu \in \mathbf{F}$, satisfy*

$$G^\nu(k/2) = 0, |k| < N, \ k \in \mathbb{Z}^d, \nu \in \mathbf{F} \tag{9}$$

then the wavelet functions $\psi^\mu, \mu \in \mathbf{F}$ have N order trigonometric vanishing moments.

From lemma 3.1, a sufficient condition of wavelets with trigonometric vanishing moments is achieved. We need solve $H(w)$ and $G^{\mu}, \mu \in \mathbf{F}$ satisfying (7), (8) and (9) in order to the construction of d-dimension orthonormal wavelets with N order trigonometric vanishing moments. Due to a large of unknown parameters, there are many difficulties in directly solving (7), (8) and (9). Under the condition of (7) and (8), we establish an equivalent condition of (9) which is imposed on the scale symbol $H(w)$ in the following theorem.

Theorem 3.2. *Let trigonometric polynomial functions $H(w)$ and $G^{\nu}(w), \nu \in \mathbf{F}$ satisfy respectively* (7)*, and* (8)*,then* (9) *is equivalent to*

$$E^{k/2}H(\pi\tau) = 0, \tau \in \mathbf{F}, |k| < N, \ k \in \mathbb{Z}^d. \tag{10}$$

We will prove this theorem after the following lemma.

Lemma 3.2. *Let $n \times n$ matrix B satisfy $BB^* = B^*B$. If the elements of the r-th row of B except for the diagonal element are all zeros, then the elements of the r-th column of B except for the diagonal element are also all zeros.*

Proof. Suppose $B = (a_{i,j})_{1 \le i,j \le n}$ and $B^* = (b_{i,j})_{1 \le i,j \le n}$. Obviously, $b_{i,j} = \overline{a}_{j,i}$. The r-th diagonal element of BB^* is $\sum_{j=1}^{n} a_{r,j} b_{j,r} = \sum_{j=1}^{n} a_{r,j} \overline{a}_{j,r}$, and the element of the r-th diagonal element of B^*B is $\sum_{k=1}^{n} b_{r,k} b_{k,r} = \sum\limits_{k=1}^{n} \overline{a}_{k,r} a_{k,r} = \sum_{k=1}^{n} |a_{k,r}|^2$. For $a_{r,j} = 0, r \neq j$, we have $|a_{r,r}|^2 = \sum_{k=1}^{n} |a_{k,r}|^2$. Then

$$\sum_{k \neq r, 1 \le k \le n} |a_{k,r}|^2 = 0 \Longrightarrow a_{k,r} = 0, \qquad k \neq r, 1 \le k \le n.$$

So the conclusion is true. □

For convenience, denote $G^0(w) = H(w)$. Let $2^d \times 2^d$ matrix A be

$$A = [G^{\mu}(w + \tau\pi)]_{\mu,\tau \in \mathbf{E}} \tag{11}$$

where μ and τ are taken strictly according to the order of the elements of E, the subscript μ and superscript τ show the row change and column change respectively.

Proof of Theorem 3.2. From (7) and (8), $AA^* = I$, where A is defined by (11) and A^* denotes the conjugate transpose of A. So A is a unitary matrix and $A^*A = I$. From lemma 3.2, it is obvious that the theorem 3.2 holds. □

Using the equivalent condition of the theorem 3.2, we can firstly solve the scaling symbol $H(w)$ satisfying (7) and (10), independent of $G^{\nu}(w), \nu \in \mathbf{F}$. Then we can obtain $G^{\nu}(w), \nu \in \mathbf{F}$, satisfying (8) by other method such as matrix extension in [7]. After the computation of the scaling filter symbol $H(w)$ and wavelet filter symbols $G^{\nu}(w), \nu \in \mathbf{F}$, define the compactly supported scaling function ϕ as follows

$$\hat{\phi}(w) = \prod_{j=1}^{\infty} H(2^{-j}w), \qquad w \in \mathbb{R}^d \tag{12}$$

and the corresponding wavelet functions $\psi^\mu, \mu \in \mathbf{F}$, are defined by (6). From lemma 3.1 and theorem 3.2, such wavelet functions $\psi^\mu, \mu \in \mathbf{F}$, have N order trigonometric vanishing moment. Our proposed construction method is practical and flexible.

Additionally, we put forward another equivalent form of (10) for computation easy.

Lemma 3.3. *Let the trigonometric polynomial function* $H(w) = \frac{1}{2^d} \sum\limits_{k\in\mathbb{Z}^d} h_k e^{-iwk}$, $w \in \mathbb{R}^d$, *satisfy* $H(0) = 1$. *Then for* $\mu \in \mathbb{Z}^d$, *the following conditions are equivalent:*
(a) $E^{\mu/2}H(\pi\tau) = 0,$ *for all* $\tau \in \mathbf{F}$.
(b) $\sum_{l\in\mathbb{Z}^d} h_{2l+\eta} e^{-i(2l+\eta)\cdot\frac{\mu}{2}} = \sum_{l\in\mathbb{Z}^d} h_{2l} e^{-il\cdot\mu},$ *for all* $\eta \in \mathbf{E}$.

Proof. An element $k \in \mathbb{Z}^d$ can be written uniquely as $2l+\gamma$ with $l \in \mathbb{Z}^d$ and $\gamma \in \mathbf{F}$.

$$E^{\mu/2}H(\pi\tau) = H(\mu/2 + \pi\tau) = \tfrac{1}{2^d}\textstyle\sum_{k\in\mathbb{Z}^d} h_k e^{-ik\pi\tau} e^{-ik\frac{\mu}{2}}$$
$$= \frac{1}{2^d}\sum_{\gamma\in\mathbf{E}}\sum_{l\in\mathbb{Z}^d} h_{2l+\gamma} e^{-i2l\pi\tau} e^{-i\pi\tau\gamma} e^{-i(2l+\gamma)\frac{\mu}{2}} = \frac{1}{2^d}\sum_{\gamma\in\mathbf{E}} e^{-i\pi\tau\gamma} e^{-i\gamma\frac{\mu}{2}}$$
$$\times \sum_{l\in\mathbb{Z}^d} h_{2l+\gamma} e^{-il\mu}$$

Hence $E^{\mu/2}H(\pi\tau) = \frac{1}{2^d}\sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i(\pi\tau+\frac{\mu}{2})\gamma}$ where $b(\gamma) = \sum_{l\in\mathbb{Z}^d} h_{2l+\gamma} e^{-il\mu}$.

The condition (a) implies
$$\sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i(\pi\tau+\frac{\mu}{2})\gamma} = 0, \forall \tau \in \mathbf{F}.$$

Let $\eta \in \mathbf{E}$,then
$$\sum_{\tau\in\mathbf{E}} e^{i\pi\eta\tau} \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i(\pi\tau+\frac{\mu}{2})\gamma} = \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i\frac{\mu}{2}\gamma} \sum_{\tau\in\mathbf{E}} e^{i\pi(\eta-\gamma)\tau}.$$

Since
$$\sum_{\tau\in\mathbf{E}} e^{i\pi(\eta-\gamma)\tau} = \begin{cases} 2^d, & \eta = \gamma \\ 0, & \eta \neq \gamma \end{cases}$$

Therefore,
$$\sum_{\tau\in\mathbf{E}} e^{i\pi\eta\tau} \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i(\pi\tau+\frac{\mu}{2})\gamma} = 2^d b(\eta) e^{-i\frac{\mu}{2}\eta}.$$

On the other hand,
$$\begin{aligned}\sum_{\tau\in\mathbf{E}} e^{i\pi\eta\tau} \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i(\pi\tau+\frac{\mu}{2})\gamma} &= \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i\frac{\mu}{2}\gamma} + \sum_{\tau\in\mathbf{F}} e^{i\pi\eta\tau} \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i(\pi\tau+\frac{\mu}{2})\gamma} \\ &= \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i\frac{\mu}{2}\gamma}\end{aligned}$$

So we have
$$2^d b(\eta) e^{-i\frac{\mu}{2}\eta} = \sum_{\gamma\in\mathbf{E}} b(\gamma) e^{-i\frac{\mu}{2}\gamma}, \text{for all } \eta \in \mathbf{E}.$$

Thereby,
$$2^d b(\eta) e^{-i\frac{\mu}{2}\eta} = 2^d b(0) e^{-i\frac{\mu}{2}0} = 2^d b(0).$$

So $\sum_{l\in\mathbb{Z}^d} h_{2l+\eta} e^{-i(2l+\eta)\cdot\frac{\mu}{2}} = \sum_{l\in\mathbb{Z}^d} h_{2l} e^{-il\cdot\mu}, \forall\ \eta \in \mathbf{E}$. $(a) \Longrightarrow (b)$ holds.

Conversely, from the condition (b), we have $b(\eta)e^{-i\frac{\mu}{2}\eta} = b(0), \forall \eta \in \mathbf{E}$. Hence, there exists

$$E^{\mu/2}H(\pi\tau) = H(\mu/2 + \pi\tau) = \frac{1}{2^d}\sum_{\gamma\in\mathbf{E}} b(\gamma)e^{-i(\pi\tau+\frac{\mu}{2})\gamma}$$
$$= \frac{1}{2^d}\sum_{\gamma\in\mathbf{E}} b(0)e^{-i\pi\tau\gamma} = \frac{1}{2^d}b(0)\sum_{\gamma\in\mathbf{E}} e^{-i\pi\tau\gamma} = \frac{1}{2^d}b(0)\sum_{\gamma\in\mathbf{E}}(-1)^{\tau\gamma}$$

In addition,

$$\sum_{\gamma\in\mathbf{E}}(-1)^{\tau\gamma} = \sum_{\gamma\in\mathbf{E}} e^{-i\pi\tau\gamma} = 0, \forall\ \tau \in \mathbf{F}$$

So $E^{\mu/2}H(\pi\tau) = 0, \forall\ \tau \in \mathbf{F}$ holds. □

On the other hand, it is well known that $\sum\limits_{\tau\in\mathbf{E}} |H(\pi\tau + w)|^2 = 1$ equals to $\sum\limits_{l\in\mathbb{Z}^d} h_l h_{l+2m} = 2^d\delta_{m,0}, m \in \mathbb{Z}^d$.

Based on the lemma 3.3 and the above formulation, $H(w)$ can be solved. But (12) is the only possible candidate for the compactly supported scaling function corresponding to the trigonometric polynomial $H(w)$ constructed. Therefore we should check whether ϕ satisfies some basic requirements for a scaling function, which are $\phi \in L^2(\mathbb{R}^d)$ and the orthonormality of the shifts of the scaling function ϕ.

If the trigonometric polynomial $H(w)$ satisfies (7), then the corresponding scaling function $\phi \in L^2(\mathbb{R}^d)$ can be guaranteed. But it is incapable of making sure the orthonormality of $\{\phi(x-k), x \in \mathbb{R}^d, k \in \mathbb{Z}^d\}$. So the orthonormal filter symbol $H(w)$ is not always coming into being the orthonormal scaling function.

The sufficient and necessary condition of orthonormal wavelets is othonormal-ityof the shifts of the scaling function ϕ. But it need additional conditions imposed on the scale symbol $H(w)$. Before we go into conditions on $H(w)$, which ensure that ϕ is orthonormal, it is interesting to know what the functions $\psi^\mu, \mu \in \mathbf{F}$, defined by (6) would be even if ϕ is not orthonormal.

For the case of 1-dimension, $\psi(x) = \frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^n\overline{h_{1-n}}\phi(2x-n)$ when $H(w)$ is orthonormal. Even if the shifts $\{\phi(x-k), x \in \mathbb{R}^d, k \in \mathbb{Z}^d\}$ of the scaling function ϕ are not othonormal, $\psi(x)$ at least can constitute a tight frame of $L^2(\mathbb{R})$ in [4]. Through the similar derivation of 1-dimension case, we generalize this conclusion for d-dimension on the condition of $H(w)$ and $G^\mu(w), \mu \in \mathbf{F}$, satisfying (7) and (8).

The frame is provided with practical signification, which can also be used to accomplish series expansion for any function belonging to $L^2(\mathbb{R}^d)$ and the approximation of such expansion is numerical stability. In detail, a little disturbance of the function will only result in a little disturbance of the coefficients and vice versa, which is just the equivalence between the L^2-norm of the function and the l^2-norm of the expanding coefficients. At this point, the frame is consistent with *Riesz* basis and orthonormal basis. But the distinctness among them is that the elements involved in the frame are correlative, which make the information of the expanding coefficients redundant. Nevertheless, in some practical application fields, such redundancy would be beneficial.

Denote

$$\phi_{j,k} = 2^{\frac{jd}{2}}\phi(2^j x - k), \quad \psi^{\mu}_{j,k} = 2^{\frac{jd}{2}}\psi^{\mu}(2^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d, \mu \in \mathbf{F}.$$

Theorem 3.3. *Let trigonometric polynomials $H(w)$ and $G^{\nu}(w), \nu \in \mathbf{F}, w \in \mathbb{R}^d$ satisfy* (7) *and* (8) *respectively, and let ϕ and $\psi^{\mu}, \mu \in \mathbf{F}$ be the compactly supported functions defined by* (12) *and* (6)*. Then for all $f \in L^2(\mathbb{R}^d)$,there exists*

$$\sum_{\mu \in \mathbf{F}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} | < f, \psi^{\mu}_{j,k} > |^2 = ||f||^2 \tag{13}$$

i.e.$\{\psi^{\mu}_{j,k}, j \in \mathbb{Z}, k \in \mathbb{Z}^d, \mu \in \mathbf{F}\}$ *constitute a tight frame of* $L^2(\mathbb{R}^d)$.

From theorem 3.3, we already have a tight frame of $L^2(\mathbb{R}^d)$ without any extra conditions on $H(w)$ except for satisfying (7) and (10),

Next we will discuss what additional conditions are necessary to make ensure that the scaling function ϕ is orthonormal shifts. W. Lawton [6] gives a sufficient and necessary condition to check whether the refinable function ϕ have orthonormal shifts. Here we use this condition to check the orthonormality of the shifts of our constructed function ϕ.

Let a finite sequence $h = \{h_k\}_{k \in \mathbb{Z}^d}$ supported in $[0, J-1]^d, J > 0$, whose symbol is $H(w)$. Define

$$B = (2^d h^{au}_{2p-q})_{p,q \in [-J+1, J-1]^d} \tag{14}$$

where the sequence h^{au} is called the autocorrelation of the sequence h and defined as follows:

$$h^{au}_k = \sum_{l \in \mathbb{Z}^d} h_{k-l}\overline{h}_{-l}, \qquad k \in \mathbb{Z}^d \tag{15}$$

If $H(0) = 1$, and the refinable function ϕ generated by h belongs to $L^2(\mathbb{R}^d)$, from [6], ϕ is orthonormal if and only if the sequence δ is a unique eigenvector of the matrix B defined by (14) corresponding to the simple eigenvalue 1.

Furthermore, we generalize the matrix extension [7] to d-dimension space. Thus the corresponding wavelets are obtained. The following theorem is easily proved based on the above discuss and the MRA of $L^2(\mathbb{R})$ (see [8]).

Theorem 3.4. *Let trigonometric polynomials $H(w)$ and $G^{\nu}(w), \nu \in \mathbf{F}, w \in \mathbb{R}^d$ satisfy* (7) *and* (8) *respectively,and for nonnegative integer N, $H(w)$ satisfies* (10)*. If the compactly supported function ϕ defined by* (12) *is orthonormal. Then the wavelets $\psi^{\mu}, \mu \in \mathbf{F}$ defined by* (6) *have N order trigonometric vanishing moments and $\{\psi^{\mu}_{j,k}, j \in \mathbb{Z}, k \in \mathbb{Z}^d, \mu \in \mathbf{F}\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$.*

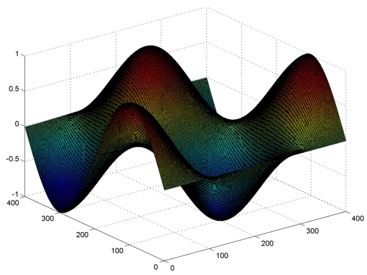

FIGURE 1. The original figure of $sinx_1sinx_2$

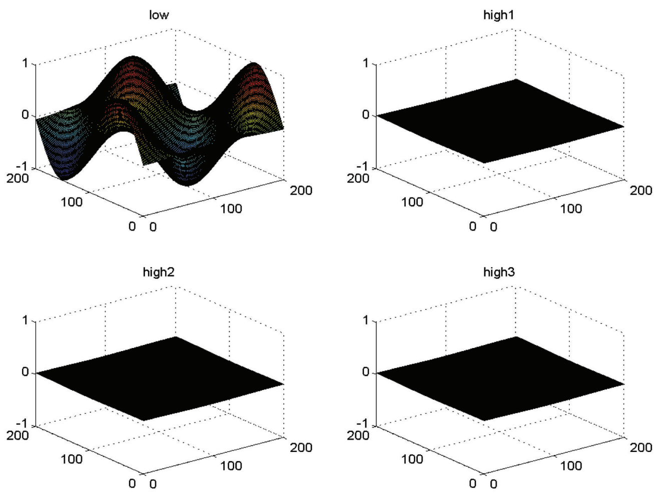

FIGURE 2. The figure of the wavelet decomposition of $f(x) = sinx_1sinx_2$

4. Numerical Example

Given $N = 2, 3$, we construct bivariate compactly supported non-tensor product orthonormal wavelets with N order trigonometric vanishing moments based on the above algorithm. The corresponding filters that we constructed are listed in Appendix. In the following an numerical example is presented to validate the trigonometric vanishing moment property of the constructed wavelets.

Let $f(x) = f(x_1, x_2) = sinx_1sinx_2$. First make a discrete process for $f(x)$. Then decompose the discrete $f(x)$ using our constructed wavelets with 3 trigonometric vanishing moments. The original figure of $f(x)$ and the figure of wavelet decomposition are given in figure 1 and figure 2 respectively. From figure 2 it is clear that the high frequencies of $f(x)$ are almost zeros, which is determined by the order of trigonometric vanishing moments.

5. Acknowledgements

The authors are grateful acknowledge Yu-Jing Guan for his useful discussion. This work is supported by the National Natural Science Foundation of China (60321002).

Appendix

For N=2: the low frequency H is

$$\left[\ A_2, B_2\ \right]$$

where

$$A_2 = \begin{bmatrix} 0.000877068502070000 & -0.00217017555027750 & -0.00160256955152750 \\ -0.00217017551704250 & 0.00446449664686000 & 0.00497124245399000 \\ -0.00160256950883500 & 0.00497124236596250 & 0.00869834985493750 \\ 0.00993380679413000 & -0.0204902519736525 & -0.0328592454723625 \\ 0.0177477598035825 & -0.0390325921033750 & -0.0565521908910750 \\ 0.00925857226745500 & -0.0202053237481550 & -0.0215681033651525 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 0.00993380688379500 & 0.0177477598463825 & 0.00925857221062500 \\ -0.0204902520504250 & -0.0390325922237225 & -0.0202053236719875 \\ -0.0328592454946325 & -0.0565521909327750 & -0.0215681032553600 \\ 0.0990133898600425 & 0.176474322895575 & 0.0750252470885100 \\ 0.176474322829218 & 0.321239655148145 & 0.144993330308675 \\ 0.0750252471651375 & 0.144993330361885 & 0.0778642975881325 \end{bmatrix}$$

the according high frequency filter $G^{(0,1)}$:

$$\left[\ A_2^{(0,1)}, B_2^{(0,1)}\ \right]$$

where

$$A_2^{(0,1)} = \begin{bmatrix} 0 & 0 & 0.00115895898017562 \\ 0 & 0 & -0.00286767156510306 \\ 0.0232400686562530 & 0.0194049573093637 & 0.0197255013320486 \\ -0.0499333461371130 & -0.0415114250914843 & -0.0336810400578003 \\ 0.0331676197772188 & 0.0198257538275190 & 0.00672983173232021 \\ -0.00377744451366267 & 0.00952362785687292 & 0.137759846989406 \\ -0.140867172341056 & -0.00903378390012367 & 0 \\ -0.0980902399371456 & -0.0243986686292091 & 0 \end{bmatrix}$$

and

$$B_2^{(0,1)} = \begin{bmatrix} -0.00286767160901982 & -0.00114093124812466 & 0.0107098277591408 \\ 0.00589938923656547 & 0.00415229141552820 & -0.0221041614680812 \\ 0.0224376140096577 & 0.0362826223006250 & 0.0254207290294945 \\ -0.0611084782654006 & -0.0836749554255576 & -0.0245975332921892 \\ 0.0785772415558306 & -0.0881703240078017 & 0.239523468582971 \\ -0.267440921656210 & 0.108470183105057 & 0.153259078849674 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the according high frequency filter $G^{(1,0)}$:

$$\left[\; A_2^{(1,0)}, B_2^{(1,0)} \; \right]$$

where

$$A_2^{(1,0)} = \begin{bmatrix} -0.00418367167599627 & 0.00274989339942355 & 0.00470186679707548 \\ 0.00860667155090754 & -0.00486498126847825 & -0.00977752356726717 \\ 0.0315305834135352 & -0.0226968594109252 & -0.0413147922667280 \\ -0.0922346339059928 & 0.0698749036239389 & 0.0795775265427353 \\ -0.118050062204540 & 0.0357086820868361 & 0.0242711625840596 \\ -0.0698354941072250 & -0.0429297840800764 & 0.00121370379275128 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$B_2^{(1,0)} = \begin{bmatrix} 0 & 0 & 0.00169081558842072 \\ 0 & 0 & -0.00418367161192572 \\ 0.0214581504037808 & 0.0891485253457010 & 0.125375417747959 \\ -0.0369870231948183 & -0.185992252041450 & -0.268609131522267 \\ 0.0259558227120533 & 0.112208320729640 & 0.217681409922140 \\ -0.0183151193075948 & 0.0516903387289760 & 0.00541826615540679 \\ -0.0191061362661868 & -0.0827098967736900 & 0 \\ 0.00663966282013075 & -0.0332239854720385 & 0 \end{bmatrix}$$

the according high frequency filter $G^{(1,1)}$:

$$\left[\; A_2^{(1,1)}, B_2^{(1,1)} \; \right]$$

where

$$A_2^{(1,1)} = \begin{bmatrix} 0 & 0 & -0.000435000892690504 \\ 0 & 0 & 0.00107634498899520 \\ -0.0133689968687221 & 0.0180659030214254 & 0.0359062140148005 \\ 0.0325728063788420 & -0.0366941425919970 & -0.0842543794721133 \\ -0.0000359525928575965 & 0.0180368366377682 & -0.0155410753412477 \\ -0.0498000453140100 & 0.0266824945690023 & 0.124655750027269 \\ 0.198098462178410 & -0.0532242431472339 & 0 \\ 0.0568425500394516 & -0.0371584022445480 & 0 \end{bmatrix}$$

and

$$B_2^{(1,1)} = \begin{bmatrix} 0.00107634500547880 & 0.00246395592796380 & -0.00905689124923250 \\ -0.00221426265133720 & -0.00659560115914500 & 0.0186588515823477 \\ 0.00152634915830774 & -0.0283531646738812 & -0.0240326834819606 \\ -0.000160425218156222 & 0.0770481698281840 & 0.0164197093731460 \\ 0.139024776016093 & -0.0229278286852003 & -0.246489256613581 \\ -0.224365982484507 & 0.165613926931721 & -0.0791089408851653 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For N=3: the low frequency H is

$$\begin{bmatrix} A_3, B_3 \end{bmatrix}$$

where

$$A_3 = \left[\begin{matrix} 0.001050692925 & -0.00137842057500000 & 0.00139268210000000 \\ -0.00137842235000000 & 0.000692282175000000 & -0.00195368947500000 \\ 0.00139268157500000 & -0.00195368792500000 & -0.0000792107125000000 \\ 0.000256387700000000 & 0.000689426350000000 & 0.00246802950000000 \\ -0.00160885055000000 & 0.00324114650000000 & -0.00182845547500000 \\ -0.00227104322500000 & 0.00155689295000000 & 0.00358252100000000 \\ 0.00425046050000000 & -0.00554880800000000 & -0.00681163550000000 \\ 0.00697939625000000 & -0.0109051725000000 & -0.0129694105000000 \\ 0.00313570675000000 & -0.0127158987500000 & 0.00279534200000000 \\ 0.00463437275000000 & -0.0103890985000000 & 0.00434127150000000 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.000256384350000000 & -0.00160884990000000 \\ 0.000689427600000000 & 0.00324114825000000 \\ 0.00246803027500000 & -0.00182845650000000 \\ 0.000256788825000000 & -0.00459710175000000 \\ -0.00459710275000000 & -0.000462449500000000 \\ -0.0100955242500000 & 0.00456146450000000 \\ 0.0136214717500000 & 0.0135751950000000 \\ 0.0412290350000000 & 0.00137962290000000 \\ 0.0307611950000000 & -0.0111617912500000 \\ 0.0104302502500000 & -0.00607148625000000 \end{matrix}\right]$$

and

$$B_3 = \left[\begin{matrix} -0.00227104110000000 & 0.00425045850000000 & 0.00697939675000000 \\ 0.00155689372500000 & -0.00554880775000000 & -0.0109051757500000 \\ 0.00358251875000000 & -0.00681163400000000 & -0.0129694082500000 \\ -0.0100955255000000 & 0.0136214702500000 & 0.0412290375000000 \\ 0.00456146675000000 & 0.0135751960000000 & 0.00137962135000000 \\ 0.0298400175000000 & -0.0238858877500000 & -0.0910682975000000 \\ -0.0238858880000000 & -0.0211511487500000 & 0.0155957102500000 \\ -0.0910683000000000 & 0.0155957120000000 & 0.204004107500000 \\ -0.0667274600000000 & 0.00540489375000000 & 0.186497255000000 \\ -0.0149734900000000 & -0.00451472125000000 & 0.0542229025000000 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.00313570775000000 & 0.00463437175000000 \\ -0.0127158972500000 & -0.0103890972500000 \\ 0.00279534225000000 & 0.00434126950000000 \\ 0.0307611925000000 & 0.0104302515000000 \\ -0.0111617927500000 & -0.00607148425000000 \\ -0.0667274575000000 & -0.0149734922500000 \\ 0.00540489400000000 & -0.00451472050000000 \\ 0.186497252500000 & 0.0542229025000000 \\ 0.252354425000000 & 0.114713485000000 \\ 0.114713485000000 & 0.0738123575000000 \end{matrix}\right]$$

the according high frequency filter $G^{(0,1)}$ is:

$$\left[\ A_3^{(0,1)}, B_3^{(0,1)}\ \right]$$

where

$$A_3^{(0,1)} = \begin{bmatrix} 0 & 0 & 0.00210840782875476 & -0.00276605339437938 & 0.00510130627173317 \\ 0 & 0 & -0.00276605695624201 & 0.00138919100219254 & -0.00694654982345029 \\ 0.0210640847478962 & 0.0239720546536162 & 0.0131150873136822 & -0.0106968956519990 & 0.00904161115774829 \\ -0.0464101621157653 & -0.0448324338217457 & -0.0320238453236112 & -0.000441982968770556 & 0.0116405843638888 \\ 0.0464162170281967 & 0.0342848595744440 & 0.0441620346722871 & 0.0398536934610097 & -0.0186153897889095 \\ -0.0148847999076046 & 0.00558033305273670 & -0.0338041163047813 & -0.0618297001230756 & -0.000633047687830840 \\ 0.00814260818561114 & -0.0467154701828439 & 0.0208972745311624 & 0.0602966585303838 & -0.0184132686843876 \\ -0.0122524301445535 & -0.0277486433152704 & 0.00819210561724972 & 0.00453166517810013 & 0.00233666403745157 \\ 0.00802790788749712 & 0.0559304402919023 & 0.0143104249569804 & -0.0376787773122579 & -0.0611862857745323 \\ 0.00129227299604935 & 0.0185563991312259 & 0.0114159327303982 & -0.0846968836587366 & 0.102230930632226 \\ -0.206251674975683 & 0.0691297388438068 & 0 & 0 & 0 \\ -0.193077118358503 & 0.0662377517001263 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B_3^{(0,1)} = \begin{bmatrix} -0.00251162793698928 & -0.00160582481643785 & -0.00211208163645764 & 0.00334974713300377 & 0.00833610915872498 \\ 0.00290326168142947 & 0.00409726174582719 & 0.00369236672183579 & -0.00168486130951536 & -0.0192392800184229 \\ -0.0169691525097504 & 0.00404201086011464 & 0.0274648207425377 & 0.00146510165211751 & 0.0167474491346390 \\ 0.0257533412886414 & -0.0135314398683656 & -0.0420702015213052 & -0.0273373179241198 & -0.0212388943774148 \\ -0.0288626918641360 & -0.0198264224147336 & 0.0136726789155425 & 0.0211677460966716 & 0.0458074455546473 \\ 0.00196291399059760 & 0.0505422469291780 & 0.0633784737497883 & -0.00300918041169930 & -0.0236528277884549 \\ 0.0229723914952898 & -0.0159993737401931 & -0.0839929469402634 & 0.00344015453596420 & -0.0107884246449990 \\ 0.0268533934244109 & -0.000404415328536778 & -0.0428119683006598 & -0.0123470593739111 & 0.0147725523291714 \\ -0.00193214541579107 & 0.0519817785838869 & 0.00215710728953622 & -0.0240772304498232 & 0.175450513818964 \\ 0.0489520963648001 & -0.127552161343852 & -0.00636118519334368 & -0.0000172362627077451 & 0.164324582631381 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the according high frequency filter $G^{(1,0)}$:

$$\left[\ A_3^{(1,0)}, B_3^{(1,0)}\ \right]$$

where

$$A_3^{(1,0)} = \left[\begin{matrix} 0 & 0 & 0.00527501632690025 \\ 0 & 0 & -0.00692038580312525 \\ 0.0430562982764235 & 0.0722381944359635 & 0.0453505977199668 \\ -0.0989103424833070 & -0.129100408313870 & -0.110105039768344 \\ 0.0668934862281779 & 0.0987500468818425 & 0.0934888354547117 \\ -0.00588318033685025 & 0.0292508074529420 & -0.000349213984559003 \\ -0.0243240357491118 & -0.181905200127919 & 0.0418438381667381 \\ 0.0430266661144508 & 0.0106705182233941 & -0.0560909736703448 \\ 0.00296329023210645 & 0.0367078054522158 & 0.0961414981891572 \\ 0.0725077035388635 & -0.0459545384210097 & 0.0178594039299865 \\ 0.0301893868652180 & -0.0504379463992989 & 0 \\ 0.0533163698587758 & -0.0400877033020673 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} -0.00692037689171673 & 0.0143677147287506 \\ 0.00347561089358910 & -0.0194848799859822 \\ 0.00801511296858635 & 0.0440175780522542 \\ -0.0661907094112600 & -0.0376050794668453 \\ 0.0747452316247861 & 0.00982383934527517 \\ -0.0446881750163803 & 0.0354779660428747 \\ 0.0925469247517093 & -0.114456588994710 \\ -0.0415160281163310 & 0.0227697014669532 \\ -0.0559230352386463 & 0.0607176374983685 \\ 0.0115554018164617 & -0.0396891818500721 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right]$$

and

$$B_3^{(1,0)} = \left[\begin{matrix} -0.00838916560067383 & 0.00214620015224831 & -0.0101884144998694 \\ 0.00832101688545020 & 0.00197114509328672 & 0.0129506154039823 \\ -0.0195083882136818 & -0.0154883109392574 & -0.00305671986155127 \\ 0.0280373673851783 & 0.0224757608820818 & -0.00000756923044740959 \\ 0.00926564511043194 & -0.0591162021033808 & 0.0365382965870718 \\ -0.0214335324120101 & 0.0473008267916477 & -0.0523073283567794 \\ -0.0463560021493588 & 0.0561947145729417 & 0.0253719840027164 \\ 0.0575023135618419 & -0.0290097964681611 & 0.0186544213384738 \\ -0.0281928196286481 & -0.0680297108965943 & 0.0488977765058992 \\ 0.0242942780057384 & 0.0288790436164232 & -0.00434344578020062 \\ 0.0301893868652180 & -0.0504379463992989 & 0 \\ 0 & 0 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.00989630705822832 & 0.0201798532694428 \\ -0.00496345833052713 & -0.0440157637577756 \\ -0.0172234277064309 & 0.0245364382262692 \\ 0.0207626529509143 & 0.0155710585771673 \\ -0.0143124664652826 & 0.0140955645965111 \\ -0.0552466436221325 & 0.0348047014923097 \\ 0.0728430848486414 & -0.0816048144015882 \\ -0.0528568157392743 & 0.0141506515007110 \\ 0.0214290414095881 & -0.0663656994848438 \\ -0.00646421647363700 & -0.0292035934395572 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right]$$

the according high frequency filter $G^{(1,1)}$:

$$\left[\ A_3^{(1,1)}, B_3^{(1,1)}\ \right]$$

where

$$A_3^{(1,1)} = \begin{bmatrix}
0 & 0 & 0.000117374244555254 & -0.000153985117649902 & 0.00221910132792689 \\
0 & 0 & -0.000153985315937410 & 0.0000773357232891733 & -0.00292542095530328 \\
0.0165581192613890 & 0.0192771703500114 & -0.00688264884627795 & -0.0121652358804233 & 0.0191928042147786 \\
-0.0250834224646992 & -0.0228401509344430 & -0.00351453831942513 & 0.00472002971601547 & -0.00572637127319008 \\
0.0375587904487678 & 0.0437987753887888 & -0.0284995953515754 & -0.00779304103223451 & 0.0238324075469179 \\
-0.0267179685329450 & 0.0330009626043208 & 0.0394850794761708 & -0.0592245039644389 & -0.0216842582050344 \\
0.00109617310856942 & -0.00930279220410017 & 0.0868316949686014 & 0.0389316512106280 & 0.0834535265849850 \\
0.0648440384668865 & -0.00307729660865373 & 0.0468852775112540 & -0.0295526682147306 & -0.0820283152110427 \\
-0.108009329679053 & 0.0781799357487619 & -0.0637692840819434 & 0.0966001541477489 & -0.0982751213673698 \\
-0.0670890385413101 & -0.0649439483839269 & 0.120268713309136 & -0.162743091596802 & 0.167760599575571 \\
0.0891689395944439 & -0.00188578160672728 & 0 & 0 & 0 \\
0.0452606821151232 & -0.0489688158222072 & 0 & 0 & 0
\end{bmatrix}$$

and

$$B_3^{(1,1)} = \begin{bmatrix}
-0.00267852730807130 & 0.000707617420121361 & 0.00267403038060246 & -0.00127230194216365 & -0.00919792524374074 \\
0.00143663407655740 & -0.00105069814653051 & 0.000310430245042659 & 0.00411508193941002 & 0.00317149071267714 \\
0.00658053803554283 & 0.00890992585869000 & 0.0346484234976300 & -0.00140929351645181 & -0.0374351916890777 \\
-0.0165262691330678 & -0.0434796922709317 & -0.0392359562564319 & -0.0000762170152020060 & -0.00171730766476417 \\
0.0183227228402569 & 0.0139265178328942 & 0.0613361594339826 & -0.000579469780395569 & -0.0533961102368122 \\
0.0479171815501325 & -0.00219586602067572 & -0.0735708287261825 & 0.0143176725285066 & 0.102737600816209 \\
-0.0211485776565514 & -0.0576433485215772 & -0.0221614678478543 & 0.0647896358857432 & -0.0655311476616241 \\
0.0464606868315364 & 0.0311295336982669 & -0.0184089302371874 & 0.0244688620170385 & 0.0214227520262634 \\
0.0966108277321110 & -0.0471218886770148 & -0.00615051575903280 & 0.00934674967705861 & 0.00549748808290365 \\
-0.0776615355776033 & -0.0342254988622366 & 0.0576556953214200 & -0.0206149532815089 & -0.0464746259449390 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

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Directional Wavelet Analysis with Fourier-Type Bases for Image Processing

Zhen Yao, Nasir Rajpoot and Roland Wilson

Abstract. Motivated by the fact that in natural images, there is usually a presence of local strongly oriented features such as directional textures and linear discontinuities, a representation which is both well-localised in frequency and orientation is desirable to efficiently describe those oriented features. Here we introduce a family of multiscale trigonometric bases for image processing using Fourier-type constructions, namely, the multiscale directional cosine transform and the multiscale Fourier transform. We also show that by seeking an adaptive basis locally, the proposed bases are able to capture both oriented harmonics as well as discontinuities, although the complexity of such adaptiveness varies significantly. We conducted denoising experiments with the proposed bases and the results show great promise of the proposed directional wavelet bases.

Keywords. Directional wavelets, curvelets, Fourier transform, cosine transform, denoising, restoration.

1. Introduction

The application of transforms in image processing is often based on a separable construction. Rows and columns in an image are treated independently and the two-dimensional basis functions are simply tensor products of the corresponding one-dimensional functions. Such method keeps simplicity in terms of design and computation, but is not capable of capturing properly all the interesting features of an image. For example, the orthonormal separable wavelet transform [5] in higher dimensions is seriously limited in its ability to efficiently represent higher dimensional features such as lines. Furthermore, the lack of frequency selectivity remains an elusive problem with most techniques operating in the wavelet domain.

Edges and textures in an image can exist at all possible locations, orientations, and scales. The ability to efficiently analyse and describe directional patterns is thus of fundamental importance for image analysis and image compression. The idea that biological visual systems might analyse image along dimensions such as orientation, scale and frequency (ie. bandpass) dates back to the work by Hubel and Wiesel [19] in the 1960's. In the computational vision literature, the idea of analysing images along multiple orientations appears at the beginning of the seventies with the Binford-Horn's line-finder [119, 120] and later work by Granlund [121]. Many edge-based image representations have then been elaborated [97, 98] with different edge detection procedures and image approximations using jump models along these edges. To refine these models, multiscale edge representations using wavelet maxima [99] or an edge-adapted multiresolution representation [100] have also been studied. Edge based image representations with complete orthonormal families of foveal wavelets in [101] and footprints [102] have been introduced and studied to reconstruct the main image edge structures. To stabilize the edge detection, global optimization procedures have also been elaborated by Donoho [103], Shukla et al. [104] and Wakin et al. [105]. The optimal configuration of edges is then calculated with an image segmentation over dyadic squares using fast dynamic programming algorithms over quadtrees. Instead of describing the image geometry through edges, which are most often ill-defined, Le Pennec and Mallat later proposed a basis named *"bandelets"* [106] which characterises the image geometry with a geometric flow of vectors. Recently, Peyré and Mallat presented the *second generation bandelets* [122]. The decomposition is computed first by the standard wavelet transform, followed by adaptive geometric orthogonal filters. The compression results are significantly better than wavelet-based coders.

All the approaches previously discussed are *adaptive* representations, in the sense that the bases are adapted to the signal/image contents. Meanwhile, from a different heuristic principle, a number of researchers have been working on developing *fixed* directional representation bases for natural images. The idea of curvelets [64] is to represent a curve as a superposition of functions of various lengths and widths obeying the scaling law $width \approx length^2$. Several different methods were proposed to construct the curvelets. A digital implementation for the curvelet transform, more commonly referred as the *curvelet-99* was used in [88] for noise removal by Starck et al. The transform first decomposes the image into subbands, i.e., separating the object into a series of disjoint scales, using the *algorithme á trous* wavelet transform. Each scale is then analysed by means of a local windowed ridgelet [54] transform. The proposed transform is $16J+1$ times redundant, with J being the number of scales for decomposition. The same authors later proposed a combined approach with curvelets and wavelets in denoising [89]. Such joint sparse representation idea is related to the idea of *Matching Pursuit* (MP) and *Basis Pursuit* (BP) [92], and another application in image deconvolution was presented in [90].

While the redundancy certainly is a advantage in the area of image restoration, it is by no means an ideal transform for compression and other tasks. In order to construct a form of discrete curvelet frame with less redundancy, Do and Vetterli [66, 67] pioneered the *"contourlet"* transform by marrying the Laplacian pyramid and a directional filter bank. Such approach is called *"double filter bank"* structure. The Laplacian pyramid mainly is used for separating isotropic features into different resolutions, then the directional filter links the point discontinuities into linear structures. This allows contourlets to efficiently approximate a smooth contour at different scales. The double filter bank design certainly allows the contourlet to be flexibly constructed. In [111], Lu and Do developed a critically sampled contourlet transform called *"CRISP-contourlet* using a combined iterated non-separable filter bank for both multiscale and directional decomposition. A non-subsampled contourlet transform was recently proposed [110]. The Laplacian pyramid was substituted with a 2-channel non-subsampled 2D filter bank which is similar to the *á trous* wavelet expansion. However, with J levels of decomposition, it has $J + 1$ redundancy. By contrast, the 2-D *á trous* algorithm by tensor product has $3J + 1$ redundancy. The whole transform has $1 + \sum_{j=1}^{J} 2^{l_j}$, where l_j denotes the number of levels in the transform at the j-th scale. Experimental results suggest that the transform compares favourably to other existing denoising and enhancement methodds reported in literature.

The curvelets can also be conveniently constructed from a frequency tiling approach. Such idea later adopted by Candés and Donoho [65] in constructing *second generation curvelets* which do not require ridgelets. Such tight frame can be computed more efficiently than the previous curvelet-99 implementation. A recent report [8] details its implementation using unequally-spaced fast Fourier transforms (USFFT) and the wrapping of specially selected Fourier samples. Both implmentations are improved in the sense that they are conceptually simpler, faster and far less redundant. The same strategy was used in constructing a 3D curvelet transform [7] whose basis functions are planar patches. These digital implementations can be found in the `CurveLab` distribution.

However, the assumption that natural images are characterised solely by linear edges is not true. Evidently we have seen attempts to separate the image into additive ingredients [3] - usually one is textural and the other is piecewisely smooth. This suggests that there is usually a presence of local strongly oriented harmonics (textures) separated by curvilinear edges. Sparse representations which are both well-localised in frequency and orientation is desirable to efficiently describe such oriented harmonic features. Also, it would be ideal to accommodate both directional linear features as well as directional periodic textures in a unified manner according to the *"image=texture+edge"* model. In this paper, we show that directional wavelet analysis can be performed with directional trigonometric transforms localised in a multiscale framework. We introduce the *Multiscale Directional Cosine Bases* in section 2 which can efficiently represent local oriented harmonics, and with a local directional cosine packet analysis, we can accommodate both directional periodic ridges and ridgelets which is a dual basis to the ridgelet packets. In section 3, we show that directional singularities and harmonics can also be captured by the *Multiresolution Fourier transform* using a Gaussian model of its magnitude spectrum with less computational burden. Next we show some results from our denoising experiments with both transforms and compare them with other wavelet transforms in section 4. The paper concludes with a summary in section 5.

2. The Multiscale Directional Cosine Transform

Like the 2D orthonormal wavelet transform, the discrete cosine transform (DCT) in 2D is also formed by tensor product, resulting in basis functions which look like "chessboard" patterns. Therefore we need to define a directional cosine operator in order to bring the orientation parameter into the transform. Also, we will need to localise the basis spatially in order to capture local features. This section describes the construction of the Multiscale Directional Cosine Transform (MDCT).

2.1. The Directional Cosine Basis

First we define the parametric space $\Gamma = \{\gamma = (k, \vec{\theta})\}$ where $k \in [0, 2\pi), \vec{\theta} \in \mathbf{S}^{d-1}$, $\vec{\theta}$ is on the unit sphere $\mathbf{S}^{d-1}$ in dimension d which indicates orientation and k indicates the frequency. Consider a family of orthonormal trigonometric basis for $L^2([0,1])$, derived from Fourier transform $\hat{f}(\xi) = \int e^{-ix\xi} f(x)dx$.

1. $\{\sqrt{2}\sin(\frac{2k+1}{2}\pi x)\}$, $k = 0, 1, 2, 3, ...$
2. $\{\sqrt{2}\sin(k\pi x)\}$, $k = 1, 2, 3, ...$
3. $\{\sqrt{2}\cos(\frac{2k+1}{2}\pi x)\}$, $k = 0, 1, 2, 3, ...$
4. $\{1, \sqrt{2}\cos(k\pi x)\}$, $k = 1, 2, 3, ...$

We denote such a trigonometric basis as $c_k(x)$, and the corresponding transform can be written as $\langle f, c_k \rangle$. Now we define the continuous directional trigonometric transform on a multi-variate function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d$:

$$\mathcal{C}_\gamma(\mathbf{x}) = c_k(\vec{\theta} \cdot \mathbf{x}) \tag{1}$$

Since $\langle f, c_k \rangle$ is essentially a Fourier transform, we have the admissibility condition

$$K_{\mathcal{C}} = \int \frac{|\hat{\mathcal{C}}(\xi)|^2}{|\xi|^d} d\xi < \infty \tag{2}$$

and the reconstruction is

$$f = \int \langle f, \mathcal{C}_\gamma \rangle \mathcal{C}_\gamma \mu(d\gamma) \tag{3}$$

The Parseval relation holds

$$\|f\|_2^2 = \int |\langle f, \mathcal{C}_\gamma \rangle|^2 \mu(d\gamma) \tag{4}$$

For a general image representation, we choose the $c_k = \{1, \sqrt{2}\cos(k\pi x)\}$, known as the cosine II basis, which has faster decay on interval $[0, 1]$ than the Fourier transform. We then have the directional 2D basis

$$\mathcal{C}_{k,\theta} = \lambda_k \cos(\pi k(x\cos\theta + y\sin\theta)) \tag{5}$$

where $\lambda_k = \begin{cases} 1 & \text{if } k = 0 \\ \sqrt{2} & \text{if } x \neq 0. \end{cases}$

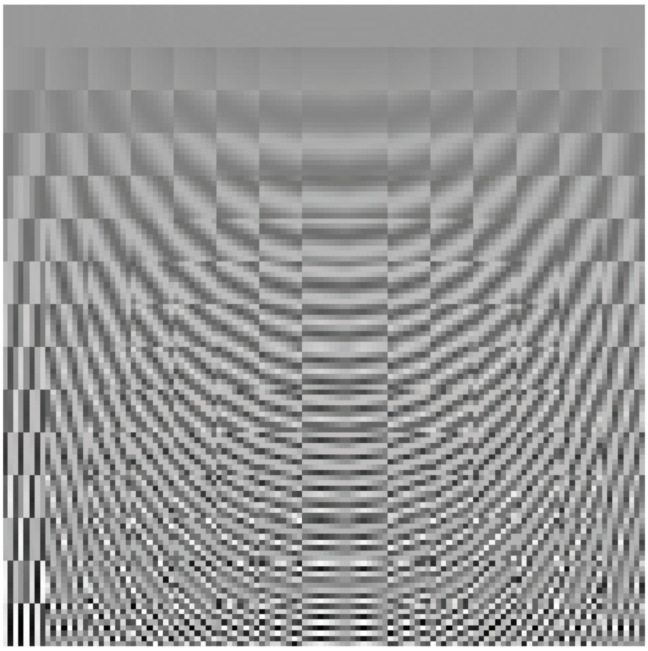


FIGURE 1. The 8×8 directional cosine basis vectors

The directional 2D continuous cosine transform is defined as

$$\begin{aligned}\mathcal{C}f(k,\theta) &= \langle f, \mathcal{C}_{k,\theta}\rangle \\ &= \int_{\mathbb{R}^2} \lambda_k f(x,y)\cos(\pi k(x\cos\theta + y\sin\theta))dxdy\end{aligned}$$

The directional cosine basis vectors are indexed by frequency k and direction θ, as can be seen in Figure 1. It is obvious that the basis vectors look similar to the Fourier basis despite the fact that the directional cosine transform is real-to-real instead of real-to-complex and its approximation error decays more rapidly than the Fourier counterpart, due to its symmetrical boundary extension.

2.2. Directional Cosine Packets

Smooth local trigonometric bases proposed by Coifman and Meyer [1] and by Malvar [2] use smooth window functions to split the signal and to fold overlapping parts back into the pieces so that the orthogonality is preserved. Therefore, the folded signal is suited for representation by a trigonometric basis. At its simplest, a typical local cosine basis function has the form :

$$\phi_{n,k}(x) = b_n(x)\cos(k + \frac{1}{2})\pi x. \tag{6}$$

where b_n is a smooth window or a *bell function*.

The usefulness of applying smooth local trigonometric bases to focus on local interesting properties of a signal is well studied, and their applications such as MP3 audio compression have been demonstrated to be successful. As we have discussed, image usually consists regions of homogenous textures separated by linear edges and contours. It is then natural to consider applying local cosine basis on Radon projected slices in order to represent both periodic patterns and some linear singularities. In this work, we use the Coifman-Wickerhauser's entropy-based best basis algorithm [45] to look for the best local cosine basis with dyadic interval. The resulting adaptive basis is similar to one possible "*ridgelet packets*" construction mentioned in [32]. We will thereafter refer to such dictionary of bases in the Radon domain as the *"directional cosine packets"*.

2.3. Multiscale Digital Implementation

For an image representation basis to be useful, the basis vectors should be localised both in space and frequency, and they should have certain orientation selectivity. More importantly, to capture patterns of interest at different scales, the basis need to be multiresolution. A prototypical MDC function has the form

$$\psi_{k,\theta,s,\mathbf{t}}(\mathbf{x}) = b\left(\frac{\mathbf{x}-\mathbf{t}}{s}\right)\mathcal{C}_{k,\theta}\left(\frac{\mathbf{x}-\mathbf{t}}{s}\right). \tag{7}$$

where k, θ, $\mathbf{t}$ and s denotes the frequency, orientation, location and scale parameters of the function respectively and $b(\cdot)$ is the smooth bell function chosen along with the sampling interval to ensure invertibility of the discrete form of the transform.

The discrete implementation of MDCT is similar to the digital curvelet-99 construction. While the discrete cosine transform and discrete Radon transform [55] are well studied in the literature, a combination of these two transforms gives us the discrete directional cosine operator. Unlike curvelet-99 which is very redundant, the multiresolution property of the MDC transform is given by the well-known decimated Laplacian pyramid [35]. The discrete MDC of a 2D vector $\mathbf{x}$, at scale s is given by

$$\mathbf{X}_s = \mathcal{C}_n(\mathbf{I} - \mathbf{G}_{s,s+1}\mathbf{G}_{s+1,s})\mathbf{x}_s. \tag{8}$$

where $\mathbf{X}_s$ denotes the transform at scale s, $\mathcal{C}_n$ is the discrete directional cosine transform operator with window size $n \times n$, $\mathbf{I}$ is the identity operator, $\mathbf{x}_s$ is the Gaussian pyramid representation of $\mathbf{x}$ at scale s

$$\mathbf{x}_s = \prod_{l=0}^{s-1} \mathbf{G}_{l+1,l}\mathbf{x}. \tag{9}$$

and $\mathbf{G}_{s,s+1}, \mathbf{G}_{s+1,s}$ are the raising and lowering operators associated with transitions between levels in the Gaussian pyramid. We certainly have the choice of using the directional cosine packets as the transform operator $\mathcal{C}_n$ by substituting the cosine transform by a cosine packet operator ψ_n, forming a semi-adaptive basis. In this way, the MDC packet basis is able to capture a wide range of directional features at different resolutions.

3. The Multiresolution Fourier Transform

The MDCT is similar to the curvelet-99 transform, only the wavelet ridge function is replaced by the cosine basis. With the local cosine analysis on Radon slices, the MDC packet bases fits well to the *“ image = edges + textures”* model.

However, the proposed bases have two limitations. The first problem is that best basis for local cosine packets has to be sought on every Radon slice, making the computation extremely expensive. Secondly, the Radon transform is a redundant transform and its inverse introduces some numerical errors. One might note that the Radon transform is directly related to the Fourier transform by the Fourier Slice Theorem, briefly stated as below:

Theorem 1. (Fourier Slice Theorem). *The 1D Fourier transform with respect to t of the projection $Rf(t,\theta)$ is equal to a central slice, at angle θ, of the 2D Fourier transform of the function $f(x,y)$, that is,*

$$\hat{Rf}(t,\theta) = \hat{f}(\xi\cos\theta, \xi\sin\theta). \tag{10}$$

where

$$\hat{f}(\xi_1,\xi_2) = \int\int f(x,y)e^{-2\pi i(x\xi_1+y\xi_2)}dxdy.$$

is the 2D Fourier transform of $f(x,y)$.

Since many discrete Radon transform are implemented via this theorem, it suggests that we may be able to perform such directional analysis using the standard Fourier transform.

3.1. The MFT Implementation

The *Multiresolution Fourier Transform* (MFT) [42, 44] has been proposed as a combination of STFT and wavelet methods, which inherits many of the desired features of both. With the windowing function $g(t)$, the transform of a function $f \in L^2(\mathbb{R})$ at position u frequency ξ and scale s is defined as:

$$Mf(u,\xi,s) = \frac{1}{\sqrt{s}}\int_{-\infty}^{+\infty} f(t)g(s(t-u))e^{-i\xi t}dt. \tag{11}$$

In effect, it is simply a stack of windowed Fourier transforms, in which the scale of the analysis window is varied systematically with the stack index. As a general image analysis tool, it has been applied in feature extraction and segmentation with music and image analysis, such as music note segmentation and extracting boundary curves in a multiresolution fashion [44]. It has also been used in texture synthesis and analysis [17] and many other areas.

The discrete implementation of MFT can take many forms. Similar to the construction of the digital MDCT described before. We build the MFT on top of the Laplacian pyramid, then on each level of the pyramid, windowed Fourier transform is performed with the same window size regardless of the scale. The discrete MFT of a 2D vector $\mathbf{x}$, at scale s is given by

$$\hat{\mathbf{x}}_s = \mathcal{F}_n(\mathbf{I} - \mathbf{G}_{s,s+1}\mathbf{G}_{s+1,s})\mathbf{x}_s. \tag{12}$$

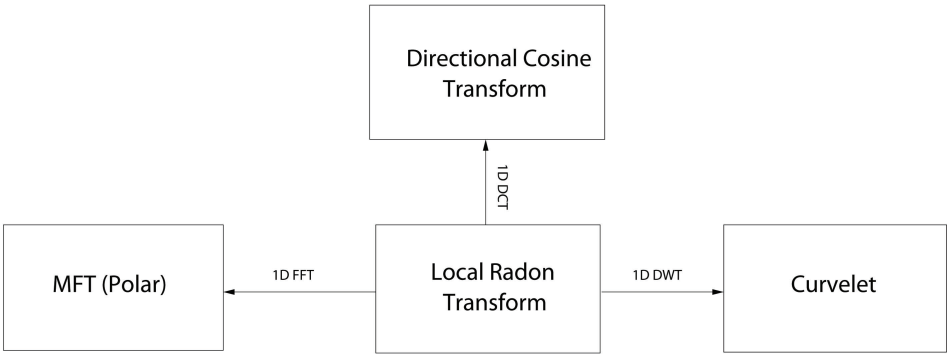


FIGURE 2. The relationships between MFT, Radon, Curvelet and directional cosine basis

where $\mathcal{F}_n$ is the discrete Fourier transform operator with window size $n \times n$. The closeness of the Burt and Adelson filter to a Gaussian function gives the pyramid virtually isotropic behavior, which can be well exploited by the high frequency resolution of the Fourier basis. The whole transform is some 5.33 times redundant if overlapping window is used.

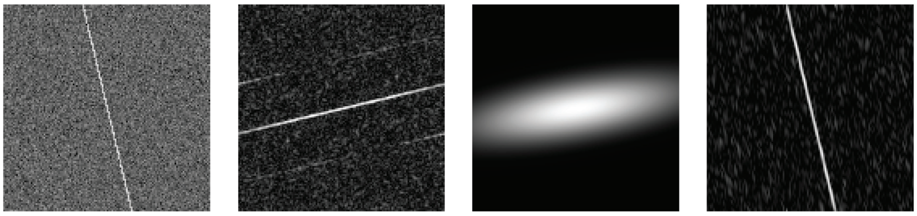
We can see that the only difference between MFT and the MDCT is that the operator used here is just a Fourier transform. In fact, the polar separablity of the Fourier transform suggests that it is also a directional trigonometric transform and Radon transform was implemented via the Fourier-slice theorem by inverse Fourier transform on Fourier polar slices. The relations between MFT, MDCT, Radon and curvelet transform can be illustrated in Figure 2. It is obvious that it requires an inverse Fourier transform and a cosine transform to convert the Fourier domain into the directional cosine domain. Although it has some advantage in approximation convergence, the extra computation is 2 times more than the conventional Fourier transform.

3.2. Gaussian Modelling of Fourier Spectrum

The Fourier basis is a natural representation for directional periodic patterns, although it decays slower than a cosine basis in terms of approximation. In order to perform some sort of *"curvelet"* analysis, we need a model for linear features in the Fourier domain. Fortunately, such model is not difficult to derive, since a line in the spatial domain will be transformed into another line in its Fourier domain perpendicular to its direction.

The magnitude intensity of the local Fourier spectrum can be modelled as a single 2D Gaussian function with its centroid fixed at the origin, which means the Gaussian is zero-mean :

$$G(\mathbf{x}) = \frac{1}{2\pi} \exp\left(\frac{-\mathbf{x}^T C^{-1} \mathbf{x}}{2}\right). \tag{13}$$



(a) (b) (c) (d)

FIGURE 3. (a) a noisy line, (b) the Fourier transform of the noisy line, (c) The Gaussian filter estimated from inertia tensor, (d) the denoised line.

The covariance matrix C of the Gaussian $G(\cdot)$ can be obtained from the inertia tensor of the spectrum.

$$C = \sum_{\tilde{\omega}} |\hat{f}(\vec{\omega})|^2 \vec{\omega}\vec{\omega}^T. \tag{14}$$

where the $\hat{f}(\vec{\omega})$ denotes the Fourier coefficients. With the covariance matrix C, the shape and the orientation of the Gaussian is determined. The centroid of the feature $\mathbf{x}_0$ can then be estimated by taking the pairwise average correlations between neighbouring coefficients in each of the horizontal and vertical directions:

$$\mathbf{x}_0 = \frac{B}{2\pi} \arg \left(\sum_{\vec{\omega}} \hat{f}(\vec{\omega} - 1) \hat{f}(\vec{\omega})^* \right). \tag{15}$$

where B is the windowing size.

The choice of using the Gaussian function is due to several reasons. First, the uncertainty principle states that the Gaussian function can achieve optimal spread in space and frequency, and it is smooth in both domains. Secondly, the shape of the 2D Gaussian function can be both isotropic and anisotropic. When the covariance matrix gives an anisotropic Gaussian distribution, this suggests that the spatial feature is a linear shape. In other cases, the Gaussian blobs will tend to be isotropic. A simple measure of the anisotropy can be obtained by performing the Principal Components Analysis (PCA) on the covariance matrix C, yielding two eignvalues λ_1 and λ_2, where $\lambda_1 \leq \lambda_2$. The measure is simply:

$$\mathcal{A} = \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right| \tag{16}$$

To test the effectiveness of our model, we use a simple linear feature with Gaussian white noise. The noisy line (see Figure 3.(a)) is transformed into its Fourier domain (see Figure 3.(b)). In order to suppress noise, we can use the $G(\mathbf{x})$ as a frequency filter in the Fourier domain. However, the problem is that the inertia

TABLE 1. The comparative image denoising results in SNR

Image	Noise (dB)	TIWP	Curvelet	MDCT	MDC Packet	MFT
barbara	0	14.75	14.59	14.89	**15.19**	14.80
	5	16.08	15.93	16.42	16.85	**17.10**
	10	18.00	17.64	18.44	18.85	**19.68**
	15	21.12	19.64	20.34	20.71	**22.46**
	20	24.92	21.41	21.88	22.10	**24.65**
lena	0	17.06	17.10	17.24	**17.75**	16.63
	5	18.96	18.82	18.99	**19.63**	19.02
	10	21.08	20.83	21.05	**21.89**	21.60
	15	23.49	23.10	23.35	24.26	**24.43**
	20	26.15	25.35	25.60	26.31	**26.94**
grain	0	12.93	13.06	13.10	**13.22**	12.85
	5	13.81	13.66	13.84	**14.29**	14.01
	10	15.87	15.01	15.51	16.18	**16.30**
	15	18.72	17.28	18.00	18.84	**19.66**
	20	22.02	20.05	20.56	21.38	**22.82**

tensor itself is easily affected by noise. Our solution is to apply thresholding on the noisy transformed data as a stage of pre-processing, in order to suppress the noise energy. The inertia tensor C then can be more reliably estimated from the thresholded data. Therefore, the resulting Gaussian frequency filter is estimated from a thresholded version of Figure 3.(b). The inversion is a clean line image without most of the noise energy. In this way, we have achieved a directional *ridgelet*-like analysis with the Fourier basis, based on the single-feature hypothesis. While the assumption is not realistic for a natural image, such library of wave packets will work well locally in a multiresolution setting. A combination of this model to the MFT allows us to analyse the signal adaptively, so that many features including contours and textures can be captured effectively.

4. Image Denoising Experiments

Good bases for reprensenting images should be able to capture important features of interest, so that the reconstruction requires as few basis functions as possible. The bases' effectiveness can be tested by performing denoising experiments by simple thresholding in the transformed domain. For the MDCT and MDC packet transform, the denoising experiments are performed in such settings:

- The Laplacian pyramid is decomposed at 5 levels of subbands.
- The window size n is chosen at 16×16, modulated with a squared cosine.
- The windows are overlapped by 50%.

The thresholding we use is a form of the universal thresholding proposed in [76], multiplied by an extra constant a, $\Theta = a\sqrt{2\log N}\sigma/1.23^L$, where $N = n^2 = 256$ here and L denotes the level of decomposition, while $L = 0$ corresponds to the highest frequency subband. For directional cosine denoising, $a = 0.08$ was found to give satisfactory result. For the directional local cosine packets, $a = 0.062$ was used. The lowpass subband is left intact.

The settings for the MFT are generally the same as for the MDCT, only with a little sophistication on estimating the filter and $a = 0.8$

1. Within each Fourier transformed block $\hat{B}$, we apply the threshold $8\sqrt{2\log 256}\sigma$ on that to obtain $\hat{B}_T$, from which the inertia tensor C will be estimated.
2. If $\mathcal{A}_C > 0.43$, which means there is a strong directional feature present, the Gaussian filter generated from C will apply on the original noisy spectrum $\hat{B}$ to obtain $\hat{B}_G$, the inverse Fourier transform is taken on $(\hat{B}_G + \hat{B}_T)/2$ as the denoised block.
3. Otherwise, we take the $\hat{B}_T$ as the denoised block.

The Fourier-type transforms are compared with two algorithms. The first is a wavelet-packet based wavelet shrinkage algorithm which is described in [15], called $\mathcal{S}$-Bayes in which the thresholding function is a modified version of the *BayesShrink* [80]. The best wavelet packet basis is sought by using the Shannon entropy function and cycle-spinning [81] is used to suppress the pseudo-Gibbs artifact. Essentially such treatment gives the translation invariance to the wavelet packet basis, which is known to be good in representing periodic signals as well as discontinuities.

The second is a modified version of curvelet. The curvelet-99 implementation reported in [88], which uses a much more redundant overcomplete wavelet frame than our MDCT and MFT, is a "specialised" transform to perform denoising task instead of general-purpose image processing. Therefore, here the local ridgelets are placed on the Laplacian pyramid as in our setting, in order to carry out a fair comparison.

We have conducted experiments on a wide range of natural images. Three of them present some typical characteristics: `barbara` contains some directional and non-directional periodic textures; `lena`, which can be regarded as one of the "curvelet-friendly" image, since it mainly consists of linear discontinuities at different scales; the `grain` image is a texture image which was considered to be very difficult to compress. It contains many directional components, however very irregular.

Table 1 gives denoising results in SNR by those five bases, where the best numbers are stressed in bold. We see that the best results are always among the MDC packet and the MFT, while the MDC packet seems to be more effective in more noisy situations. This is due to the fact that the Gaussian filters are estimated from the noise-sensitive inertia tensor. When the spectrum are dominated by the noise energy, a simple thresholding would fail to preserve the signal information. However, the MFT can be considered as the overall winner: it compares well with

Noisy, 10dB | TIWP-$\mathcal{S}$-Bayes, 18.00dB | Curvelet, 17.64dB

Directional cosine, 18.44 dB | Directional local cosine, 18.85 dB | MFT, 19.68 dB

FIGURE 4. Detailed comparative denoising results on `barbara`

other denoising methods and its computational cost is much lower than the rest of the methods. We also notice that TIWP-$\mathcal{S}$-BayesShrink sometimes outperforms at some low noise levels, since the BayesShrink tends to optimise the MSE output. However, the visual qualities of other candidates are more pleasing, preserving important directional features on the image.

A detailed head-to-head comparison is presented in Figure 4 on `barbara`. It is obvious that from the TIWP-$\mathcal{S}$-BayesShrink thresholded image, the diagonal strips are absent on the cloth in the middle, although a few of such patterns can be seen on the trousers. The curvelet is able to recover some of those directional patterns, but incomplete nonetheless. These features are restored almost completely by our proposed methods.

Since the directional cosine packets can be regarded as a generalisation of the curvelets and directional cosine bases, it is not suprising to see it gives better results than these two counterparts. However, it introduces considerable amount of extra computations, since the best basis has to be sought for each of the Radon slices. Also, the inverse Radon transform from incomplete data introduces numerical errors in the reconstruction. Although visually MFT and MDC packet both captured linear and oscillating patterns, MFT's reconstruction is much sharper and outperforms by quite a margin in SNR. On the other hand, the best-basis search ensures that the denoised image from MDC packet is smoother and almost "artifact-free". Considering the visual/statistical performance and the complexity, the MFT with Gaussian filter is the overall winner.

5. Conclusions

In this paper, we have reviewed a growing literature body on directional wavelets' construction, analysis and their applications. The contribution of this paper is to introduce sets of Fourier-type bases which have localisation in space and frequency, orientation selectivity, and employing a multiresolution pyramidal framework allowing analyses of images at different scales.

In a sense, the Fourier-type bases qualify as geometrical wavelets and share a lot of similarities with other directional wavelet bases proposed previously. But the semi-adaptiveness allows us to capture local directional texture patches and linear features at ease and we have shown that by a simple Gaussian frequency filter model of magnitude spectrum intensity, analysis of directional harmonics and linear features can be carried out much more efficiently than any other directional wavelet bases proposed to date. It can be considered as a parametric curvelet representation, or a *generalised directional wavelet packets* and the simple inertia tensor method has demonstrated to be a good substitute for the Cartesian-polar conversion. More importantly, its computational cost is a big advantage, since it does not involve a notion of Radon transform, nor the best-basis search as in some adaptive representations.

The effectiveness of the proposed bases was tested against the state-of-the-art translation-invariant wavelet packet based shrinkage method and the curvelets. The new bases demonstrated a strong potential in the experiments, outperforming the opponents by quite a margin. While producing much visually pleasant output than the wavlet packets with optimal threshold, the MDC bases seems to be able to capture a wider range of directional features than the curvelet, even without the local cosine treatment.

The denoising experiments show the effectiveness of conducting multiresolution analysis with these bases. A wide variation on the theme is possible, for example using variable sized windows on the original image might be another possibility, or to use the *algorithme á trous* subband decomposition for better denoising results. The usage of the Gaussian frequency filter and its parameter estimation in noisy environments are still under investigation. It is our intention to put forward these bases in a general way in this work to popularise their usage in various kinds of image processing tasks.

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Unitary Systems and Wavelet Sets

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Abstract. A wavelet is a special case of a vector in a separable Hilbert space that generates a basis under the action of a system of unitary operators defined in terms of translation and dilation operations. We will describe an operator-interpolation approach to wavelet theory using the local commutant of a unitary system. This is an application of the theory of operator algebras to wavelet theory. The concrete applications to wavelet theory include results obtained using specially constructed families of wavelet sets. The main section of this paper is section 5, in which we introduce the interpolation map σ induced by a pair of wavelet sets, and give an exposition of its properties and its utility in constructing new wavelets from old. The earlier sections build up to this, establishing terminology and giving examples. The main theoretical result is the Coefficient Criterion, which is described in Section 5.2.2, and which gives a matrix valued function criterion specificing precisely when a function with frequency support contained in the union of an interpolation family of wavelet sets is in fact a wavelet. This can be used to derive Meyer's famous class of wavelets using an interpolation pair of Shannon-type wavelet sets as a starting point. Section 5.3 contains a new result on interpolation pairs of wavelet sets: a proof that every pair of sets in the generalized Journe family of wavelet sets is an interpolation pair. We will discuss some results that are due to this speaker and his former and current students. And we finish in section 6 with a discussion of some open problems on wavelets and frame-wavelets.

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Keywords. Wavelet, wavelet set, unitary system, frame.

1. Introduction

A wavelet is a special case of a vector in a separable Hilbert space that generates a basis under the action of a collection, or "system", of unitary operators defined in

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terms of translation and dilation operations. This approach to wavelet theory goes back, in particular, to earlier work of Goodman, Lee and Tang [10] in the context of multiresolution analysis. We will begin by describing the operator-interpolation approach to wavelet theory using the local commutant of a system that was worked out by the speaker and his collaborators a few years ago. This is really an abstract application of the theory of operator algebras, mainly von Neumann algebras, to wavelet theory. The concrete applications of operator-interpolation to wavelet theory include results obtained using specially constructed families of wavelet sets. In fact X. Dai and the speaker had originally developed our theory of wavelet sets [5] specifically to take advantage of their natural and elegant relationships with these wavelet unitary systems. We will also discuss some new results and open questions.

The main idea in *operator-theoretic interpolation* of wavelets (and frames) is that new wavelets can be obtained as linear combinations of known ones using *coefficients* which are *operators* (in fact, *Fourier multipliers*) in a certain class. Both the ideas and the essential computations extend naturally to more general unitary systems and *wandering vectors.* Many of the methods work for more involved systems that are important to applied harmonic analysis, such as Gabor and generalized Gabor systems, and various types of *frame* unitary systems.

1.1. Terminology

The set of all bounded linear operators on a Hilbert space H will be denoted by $B(H)$. A *bilateral shift* U on H is a unitary operator U for which there exists a closed linear subspace $E \subset H$ with the property that the family of subspaces $\{U^n E : n \in \mathbb{Z}\}$ are orthogonal and give a direct-sum decomposition of H. The subspace E is called a *complete wandering subspace* for U, and the *multiplicity* of U is defined to be the dimension of E. The *strong operator topology* on $B(H$ is the topology of pointwise convergence, and the *weak operator topology* is the weakest topology such that the vector functionals $\omega_{x,y}$ on $B(H)$ defined by $A \mapsto \langle Ax, y\rangle$, $A \in B(H)$, $x, y \in H$, are all continuous. An *algebra of operators* is a linear subspace of $B(H)$ which is closed under multiplication. An *operator algebra* is an algebra of operators which is *norm-closed*. A subset $\mathcal{S} \subset B(H)$ is called *selfadjoint* if whenever $A \in \mathcal{S}$ then also $A^* \in \mathcal{S}$. A C^*-*algebra* is a self-adjoint operator algebra. A *von Neumann algebra* is a C^*-algebra which is closed in the weak operator topology. For a unital operator algebra, it is well known that being closed in the weak operator topology is equivalent to being closed in the closed in the strong operator topology. The *commutant* of a set $\mathcal{S}$ of operators in $B(H)$ is the family of all operators in $B(H)$ that *commute* with every operator in $\mathcal{S}$. It is closed under addition and multiplication, so is an algebra. And it is clearly closed in both the weak operator topology and the strong operator topology. We use the standard *prime* notation for the commutant. So the commutant of a subset $\mathcal{S} \subset B(H)$ is denoted: $\mathcal{S}' := \{A \in B(H) : AS = SA,\ S \in \mathcal{S}\}$. The commutant of a selfadjoint set of operators is clearly a von Neumann algebra. Moreover, by a famous theorem of Fuglede every operator which commutes with a normal operator N also commutes with its adjoint N^*, and hence the commutant of any set of *normal* operators is also a von Neumann algebra. So, of particular relevance to this work, the commutant of any set of *unitary* operators is a von Neumann algebra.

1.1.1. Frames and Operators. A sequence of vectors $\{f_j\}$ in a separable Hilbert space H is a *frame* (or *frame sequence*) if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|f\|_2^2 \leq \sum_j |< f, f_j >|^2 \leq C_2\|f\|_2^2$$

for all $f \in H$. If $C_1 = C_2$ the frame is called *tight*, and if $C_1 = C_2 = 1$, $\{f_j\}$ is called a Parseval frame. (The term *normalized tight* has also been used for this (cf [14]). A vector ξ is called a $f$rame vector for a unitary system $\mathcal{U}$ if the set of vectors $\mathcal{U}\xi$ is a frame for H.

A Riesz basis for a Hilbert space is a bounded unconditional basis. Frames sequences are generalizations of Riesz bases. A number of the basic aspects of a geometric, or operator-theoretic, approach to discrete frame theory on Hilbert space arises from the fact that a frame sequence is simply an "inner" direct summand of a Riesz basis. The basic principle is that a Hilbert space frame sequence can be dilated to a Riesz basis for larger Hilbert space. We call this the *Frame Dilation Theorem*. In other words, for a given frame sequence there is a larger Hilbert space and a Riesz basis for the larger space such that the orthogonal projection from the larger space onto the smaller space compresses the Riesz basis to the frame sequence. We proved this at the beginning of [14], and used it to prove the other results [14], and subsequently to prove some applications to Hilbert C*-module theory jointly with M. Frank. We proved it first for Parseval frames, and then for general frames. (We remark that this type of dilation result for frames was also independently known and used independently by several others in different contexts.)

It is interesting to note that the Parseval frame case of the Frame Dilation Theorem can be derived easily from the purely atomic case of a well known theorem of Naimark on projection valued measures. We thank Chandler Davis for pointing this out to us at the Canadian Operator Algebras Symposium in 1999. We (Han and I) basically proved this special case of Naimark's theorem implicitely in the first section of [14] without recognizing it was a special case of Naimark's theorem, and then we proved the appropriate generalization we needed for general (non-tight) frames. Naimark's Dilation Theorem basically states that a suitable positive operator valued measure on a Hilbert space *dilates* to a projection valued measure on a larger Hilbert space. That is, there is a projection valued measure [PVM] on a larger Hilbert space such that the orthogonal projection from the larger space unto the smaller space compresses the [PVM] to the [POVM]. In the discrete (i.e. purely atomic measure) case, it can be interpreted as stating that a suitable sequence of positive operators dilates to a sequence of projections. The dilation theorem for a Parseval frame follows easily from Naimark's Theorem applied to the [POVM] obtained by replacing each vector x_i in the frame sequence with the elementary tensor operator $x_i \otimes x_i$, obtaining the atoms for a [POVM] defined on all subsets of the index set for the frame. The dilation theorem for a general (non-tight) frame does not seem to follow directly from Naimark's theorem – but it may follow from a generalization of it. We remark that some other generalizations of the frame dilation theorem have been recently worked out, notably by W. Czaja.

1.1.2. Unitary Systems, Wandering Vectors, and Frame Vectors. We define a *unitary system* to be simply a countable collection of unitary operators $\mathcal{U}$ acting on a Hilbert space H which contains the identity operator. The *interesting* unitary systems all have additional structural properties of various types. (For instance, wavelet systems and Gabor systems are both "ordered products" of two abelian groups: the dilation and translation groups in the wavelet case, and the modulation and translation groups in the Gabor case.) We will say that a vector $\psi \in H$ is *wandering* for $\mathcal{U}$ if the set

$$\mathcal{U}\psi := \{U\psi : U \in \mathcal{U}\} \tag{1}$$

is an orthonormal set, and we will call ψ a *complete wandering vector* for $\mathcal{U}$ if $\mathcal{U}\psi$ spans H. This (abstract) point of view can be useful. Write $\mathcal{W}(\mathcal{U})$ for the set of complete wandering vectors for $\mathcal{U}$. Further, a *Riesz vector* for $\mathcal{U}$ is a vector ψ such that $\mathcal{U}\psi$ is a Riesz basis for H (indexed by the elements of $\mathcal{U}$), and a *frame vector* is a vector ψ such that $\mathcal{U}\psi$ is a frame sequence for H (again using $\mathcal{U}$ as its index set), and we adopt similar terminology for Parseval frame vectors and Bessel vectors. We use $\mathcal{R}W(\mathcal{U})$, $\mathcal{F}(\mathcal{U})$, $\mathcal{P}F(\mathcal{U})$, $\mathcal{B}(\mathcal{U})$ to denote, respectively, the sets of Riesz vectors, frame vectors, Parseval frame vectors, and Bessel vectors for $\mathcal{U}$.

One of the main tools in this work is the *local commutant* of a system of unitary operators (see section 3.2). This is a natural generalization of the commutant of the system, and like the commutant it is a linear space of operators which is closed in the weak and the strong operator topologies, but unlike the commutant it is usually not selfadjoint, and is usually not closed under multiplication. It contains the commutant of the system, but can be much larger than the commutant. The local commutant of a wavelet unitary system captures all the information about the wavelet system in an essential way, and this gives the *flavor* of our approach to the subject.

1.1.3. Normalizers. If U is a unitary operator and $\mathcal{A}$ is an operator algebra, then U is said to *normalize* $\mathcal{A}$ if $U^\star \cdot \mathcal{A} \cdot U = \mathcal{A}$. In the most interesting cases of operator-theoretic interpolation: that is, for those cases that yield the strongest structural results, the relevant unitaries in the local commutant of the system normalize the commutant of the system.

1.2. Acknowledgement

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2. Wavelets

For simplicity of presentation, much of the work in this article will deal with one-dimensional wavelets, and in particular, the dyadic case. The other cases: non-dyadic wavelets and wavelets in higher dimensions, are at least notationally more complicated.

2.1. One Dimension

A *dyadic orthonormal* wavelet in one dimension is a unit vector $\psi \in L^2(\mathbb{R}, \mu)$, with μ Lebesgue measure, with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - l) : n, l \in \mathbb{Z}\} \tag{2}$$

of all integral translates of ψ followed by dilations by arbitrary integral powers of 2, is an orthonormal basis for $L^2(\mathbb{R}, \mu)$. The term *dyadic* refers to the dilation factor "2". The term *mother wavelet* is also used in the literature for ψ. Then the functions

$$\psi_{n,l} := 2^{\frac{n}{2}}\psi(2^n t - l)$$

are called elements of the wavelet basis generated by the "mother". The functions $\psi_{n,l}$ will not themselves be mother wavelets unless $n = 0$.

Let T and D be the translation (by 1) and dilation (by 2) unitary operators in $B(L^2(\mathbb{R})$ given by $(Tf)(t) = f(t-1)$ and $(Df)(t) = \sqrt{2}f(2t)$. Then

$$2^{\frac{n}{2}}\psi(2^n t - l) = (D^n T^l \psi)(t)$$

for all $n, l \in \mathbb{Z}$. Operator-theoretically, the operators T, D are *bilateral shifts* of *infinite multiplicity*. It is obvious that $L^2([0,1])$, considered as a subspace of $L^2(\mathbb{R})$, is a complete wandering subspace for T, and that $L^2([-2,-1] \cup [1,2])$ is a complete wandering subspace for D.

Let $\mathcal{U}_{D,T}$ be the unitary system defined by

$$\mathcal{U}_{D,T} = \{D^n T^l : n, l \in \mathbb{Z}\} \tag{3}$$

where D and T are the operators defined above. Then ψ is a dyadic orthonormal wavelet if and only if ψ is a complete wandering vector for the unitary system $\mathcal{U}_{D,T}$. This was our original motivation for developing the abstract unitary system theory. Write

$$\mathcal{W}(D, T) := \mathcal{W}(\mathcal{U}_{D,T}) \tag{4}$$

to denote the set of all dyadic orthonormal wavelets in one dimension.

An abstract interpretation is that, since D is a bilateral shift it has (many) complete wandering subspaces, and a wavelet for the system is a vector ψ whose translation space (that is, the closed linear span of $\{T^k : k \in \mathbb{Z}\}$ is a complete wandering subspace for D. Hence ψ must generate an orthonormal basis for the entire Hilbert space under the action of the unitary system.

In one dimension, there are non-dyadic orthonormal wavelets: i.e. wavelets for all possible dilation factors besides 2 (the dyadic case). We said "possible", because the scales $\{0, 1, -1\}$ are excluded as scales because the dilation operators they would introduce are not bilateral shifts. All other real numbers for scales yield wavelet theories. In [5, Example 4.5 (x)] a family of examples is given of three-interval wavelet sets (and hence wavelets) for all scales $d \geq 2$, and it was noted there that such a family also exists for dilation factors $1 < d \leq 2$. There is some recent (yet unpublished) work that has been done, by REU students and mentors, building on this, classifying finite-interval wavelet sets for all possible real (positive and negative scale factors). I will mention this work, in passing, in my talk.

2.2. N-Dimensions

2.2.1. Expansive Dilations. Let $1 \leq m < \infty$, and let A be an $n \times n$ real matrix which is *expansive* (equivalently, all (complex) eigenvalues have modulus > 1). By a *dilation - A regular-translation orthonormal wavelet* we mean a function $\psi \in L^2(\mathbb{R}^n)$ such that

$$\{|det(A)|^{\frac{n}{2}} \psi(A^n t - (l_1, l_2, ..., l_n)^t) : n, l \in \mathbb{Z}\} \tag{5}$$

where $t = (t_1, ..., t_n)^t$, is an orthonormal basis for $L^2(\mathbb{R}^n; m)$. (Here m is product Lebesgue measure, and the superscript "t" means transpose.)

If $A \in M_n(\mathbb{R})$ is invertible (so in particular if A is expansive), then it is very easy to verify that the operator defined by

$$(D_A f)(t) = |detA|^{\frac{1}{2}} f(At) \tag{6}$$

for $f \in L^2(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, is *unitary.* For $1 \leq i \leq n$, let T_i be the unitary operator determined by translation by 1 in the i^{th} coordinate direction. The set (5) above is then

$$\{D_A^k T_1^{l_1} \cdots T_n^{l_n} \psi : k, l_i \in \mathbb{Z}\} \tag{7}$$

If the dilation matrix A is expansive, but the translations are along some oblique lattice, then there is an invertible real $n \times n$ matrix T such that conjugation with D_T takes the entire wavelet system to a regular-translation expansive-dilation matrix. This is easily worked out, and was shown in detail in [18] in the context of working out a complete theory of unitary equivalence of wavelet systems. Hence the wavelet theories are equivalent.

2.2.2. Non-expansive Dilations. Much work has been accomplished concerning the existence of wavelets for dilation matrices A which are not expansive. Some of the original work was accomplished in the Ph.D. theses of Q. Gu and D. Speegle, when they were together finishing up at Texas A&M. Some significant additional work was accomplished by Speegle and also by others. In [18], with Ionascu and Pearcy we proved that if an nxn real invertible matrix A is not similar (in the nxn complex matrices) to a unitary matrix, then the corresponding dilation operator D_A is in fact a bilateral shift of infinite multiplicity. If a dilation matrix were to admit any type of wavelet (or frame-wavelet) theory, then it is well-known that a necessary condition would be that the corresponding dilation operator would have to be a bilateral shift of infinite multiplicity. I am happy to report that in very recent work [23], with E. Schulz, D. Speegle, and K. Taylor, we have succeeded in showing that this minimal condition is in fact sufficient: such a matrix, with regular translation lattice, admits a (perhaps infinite) tuple of functions, which collectively generates a frame-wavelet under the action of this unitary system.

3. More General Unitary Systems

3.1. Some Restrictions

We note that *most* unitary systems $\mathcal{U}$ do not have complete wandering vectors. For $\mathcal{W}(\mathcal{U})$ to be nonempty, the set $\mathcal{U}$ must be very special. It must be *countable* if it acts separably (i.e. on a separable Hilbert space), and it must be *discrete* in the strong operator topology because if $U, V \in \mathcal{U}$ and if x is a wandering vector for $\mathcal{U}$ then

$$\|U - V\| \geq \|Ux - Vx\| = \sqrt{2}$$

Certain other properties are forced on $\mathcal{U}$ by the presence of a wandering vector. (Or indeed, by the nontriviality of any of the sets $\mathcal{W}(\mathcal{U})$, $\mathcal{RW}(\mathcal{U})$, $\mathcal{F}(\mathcal{U})$, $\mathcal{PF}(\mathcal{U})$, $\mathcal{B}(\mathcal{U})$.) One purpose of [5] was to investigate such properties. Indeed, it was a matter of some surprise to us to discover that such a theory is viable even in some considerable generality. For perspective, it is useful to note that while $\mathcal{U}_{D,T}$ has complete wandering vectors, the reversed system

$$\mathcal{U}_{T,D} = \{T^l D^n : n, l \in \mathbb{Z}\}$$

fails to have a complete wandering vector. (A proof of this was given in the introduction to [5].)

3.2. The Local Commutant

3.2.1. A Special Case: The System $\mathcal{U}_{D,T}$. Computational aspects of operator theory can be introduced into the wavelet framework in an elementary way. Here is the way we originally did it: Fix a wavelet ψ and consider the set of all operators $A \in B(L^2(\mathbb{R}))$ which *commute* with the *action* of dilation and translation on ψ. That is, require

$$(A\psi)(2^n t - l) = A(\psi(2^n t - l)) \tag{8}$$

or equivalently

$$D^n T^l A\psi = AD^n T^l \psi \tag{9}$$

for all $n, l \in \mathbb{Z}$. Call this the *local commutant of the wavelet system* $\mathcal{U}_{D,T}$ *at the vector* ψ. (In our first preliminary writings and talks we called it the *point commutant* of the system.) Formally, the local commutant of the dyadic wavelet system on $L^2(\mathbb{R})$ is:

$$\mathcal{C}_\psi(\mathcal{U}_{D,T}) := \{A \in B(L^2(\mathbb{R})) : (AD^n T^l - D^n T^l A)\psi = 0, \forall n, l \in \mathbb{Z}\} \tag{10}$$

This is a linear subspace of $B(H)$ which is closed in the strong operator topology, and in the weak operator topology, and it clearly contains the *commutant* of $\{D, T\}$.

A motivating example is that if η is any other wavelet, let $V := V_\psi^\eta$ be the unitary (we call it the *interpolation unitary*) that takes the basis $\psi_{n,l}$ to the basis $\eta_{n,l}$. That is, $V\psi_{n,l} = \eta_{n,l}$ for all $n, l \in \mathbb{Z}$. Then $\eta = V\psi$, so $VD^nT^l\psi = D^nT^lV\psi$ hence $V \in \mathcal{C}_\psi(\mathcal{U}_{D,T})$.

In the case of a pair of complete wandering vectors ψ, η for a general unitary system $\mathcal{U}$, we will use the same notation V_ψ^η for the unitary that takes the vector $U\psi$ to $U\eta$ for all $U \in \mathcal{U}$.

This simple-minded idea is reversible, so for every unitary V in $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ the vector $V\psi$ is a wavelet. This correspondence between unitaries in $\mathcal{C}_\psi(D,T)$ and dyadic orthonormal wavelets is one-to-one and onto (see Proposition 3.1). This turns out to be useful, because it leads to some new formulas relating to decomposition and factorization results for wavelets, making use of the *linear* and *multiplicative* properties of $\mathcal{C}_\psi(D,T)$.

It turns out (a proof is required) that the entire local commutant of the system $\mathcal{U}_{D,T}$ at a wavelet ψ is *not* closed under multiplication, but it also turns out (also via a proof) that for *most* (and perhaps *all*) wavelets ψ the local commutant at ψ contains many noncommutative operator algebras (in fact von Neumann algebras) as subsets, and their unitary groups *parameterize* norm-arcwise-connected families of wavelets. Moreover, $\mathcal{C}_\psi(D,T)$ is closed under *left multiplication* by the commutant $\{D,T\}'$, which turns out to be an abelian nonatomic von Neumann algebra. The fact that $\mathcal{C}_\psi(D,T)$ is a *left module* under $\{D,T\}'$ leads to a method of obtaining new wavelets from old, and of obtaining connectedness results for wavelets, which we called *operator-theoretic interpolation* of wavelets in [5], (or simply *operator-interpolation*).

3.2.2. General Systems. More generally, let $\mathcal{S} \subset B(H)$ be a set of operators, where H is a separable Hilbert space, and let $x \in H$ be a nonzero vector, and *formally* define the *local commutant* of $\mathcal{S}$ at x by

$$\mathcal{C}_x(\mathcal{S}) := \{A \in B(H) : (AS - SA)x = 0, S \in \mathcal{S}\}$$

As in the wavelet case, this is a weakly and strongly closed linear subspace of $B(H)$ which contains the commutant $\mathcal{S}'$ of $\mathcal{S}$. If x is *cyclic* for $\mathcal{S}$ in the sense that span$(\mathcal{S}x)$ is dense in H, then x *separates* $\mathcal{C}_x(\mathcal{S})$ in the sense that for $S \in \mathcal{C}_x(\mathcal{S})$, we have $Sx = 0$ iff $x = 0$. Indeed, if $A \in \mathcal{C}_x(\mathcal{S})$ and if $Ax = 0$, then for any $S \in \mathcal{S}$ we have $ASx = SAx = 0$, so $A\mathcal{S}x = 0$, and hence $A = 0$.

If $A \in \mathcal{C}_x(\mathcal{S})$ and $B \in \mathcal{S}'$, let $C = BA$. Then for all $S \in \mathcal{S}$,

$$(CS - SC)x = B(AS)x - (SB)Ax = B(SA)x - (BS)Ax = 0$$

because $ASx = SAx$ since $A \in \mathcal{C}_x(\mathcal{S})$, and $SB = BS$ since $B \in \mathcal{S}'$. Hence $\mathcal{C}_x(\mathcal{S})$ is closed under left multiplication by operators in $\mathcal{S}'$. That is, $\mathcal{C}_x(\mathcal{S}$ is a *left module* over $\mathcal{S}'$.

It is interesting that, if in addition $\mathcal{S}$ is a multiplicative semigroup, then in fact $\mathcal{C}_x(\mathcal{S})$ is identical with the commutant $\mathcal{S}'$ so in this case the commutant is not a new structure. To see this, suppose $A \in \mathcal{C}_x(\mathcal{S})$. Then for each $S, T \in \mathcal{S}$ we have $ST \in \mathcal{S}$, and so

$$AS(Tx) = (ST)Ax = S(ATx) = (S)Tx$$

So since $T \in \mathcal{S}$ was arbitrary and span$(\mathcal{S}x) = H$, it follows that $AS = SA$.

Proposition 3.1. *If $\mathcal{U}$ is any unitary system for which $\mathcal{W}(\mathcal{U}) \neq \emptyset$, then for any $\psi \in \mathcal{W}(\mathcal{U})$*

$$\mathcal{W}(\mathcal{U}) = \{U\psi : U \text{ is a unitary operator in } \mathcal{C}_\psi(\mathcal{U})\}$$

and the correspondence $U \to U\psi$ is one-to-one.

Proposition 3.2. *Let $\mathcal{U}$ be a unitary system on a Hilbert space H. If ψ is a complete wandering vector for $\mathcal{U}$, then:*

(i) $\mathcal{RW}(\mathcal{U}) = \{A\psi : A$ *is an operator in* $\mathcal{C}_\psi(\mathcal{U})$ *that is invertible in* $B(H)\}$;
(ii) $\mathcal{F}(\mathcal{U}) = \{A\psi : A$ *is an operator in* $\mathcal{C}_\psi(\mathcal{U})$ *that is surjective*$\}$;
(ii) $\mathcal{PF}(\mathcal{U}) = \{A\psi : A$ *is an operator in* $\mathcal{C}_\psi(\mathcal{U})$ *that is a co-isometry*$\}$;
(ii) $\mathcal{B}(\mathcal{U}) = \{A\psi : A$ *is an operator in* $\mathcal{C}_\psi(\mathcal{U})\}$

3.3. Operator-Theoretic Interpolation

Now suppose $\mathcal{U}$ is a unitary system, such as $\mathcal{U}_{D,T}$, and suppose $\{\psi_1, \psi_2, \ldots, \psi_m\} \subset \mathcal{W}(\mathcal{U})$. (In the case of $\mathcal{U}_{D,T}$, this means that $(\psi_1, \psi_2, \ldots, \psi_n)$ is an n-tuple of wavelets.

Let $(A_1, A_2, \ldots, A_n)$ be an n-tuple of operators in the commutant $\mathcal{U}'$ of $\mathcal{U}$, and let η be the vector

$$\eta := A_1\psi_1 + A_2\psi_2 + \cdots + A_n\psi_n \ .$$

Then

$$\begin{aligned} \eta &= A_1\psi_1 + A_2 V_{\psi_1}^{\psi_2}\psi_1 + \ldots A_n V_{\psi_1}^{\psi_n}\psi_1 \\ &= (A_1 + A_2 V_{\psi_1}^{\psi_2} + \cdots + A_n V_{\psi_1}^{\psi_n})\psi_1 \ . \end{aligned} \tag{11}$$

We say that η is obtained by *operator interpolation* from $\{\psi_1, \psi_2, \ldots, \psi_m\}$. Since $\mathcal{C}_{\psi_1}(\mathcal{U})$ is a left $\mathcal{U}'$ - module, it follows that the operator

$$A := A_1 + A_2 V_{\psi_1}^{\psi_2} + \ldots A_n V_{\psi_1}^{\psi_n} \tag{12}$$

is an element of $\mathcal{C}_{\psi_1}(\mathcal{U})$. Moreover, if B is another element of $\mathcal{C}_{\psi_1}(\mathcal{U})$ such that $\eta = B\psi_1$, then $A - B \in \mathcal{C}_{\psi_1}(\mathcal{U})$ and $(A-B)\psi_1 = 0$. So since ψ_1 separates $\mathcal{C}_{\psi_1}(\mathcal{U})$ it follows that $A = B$. Thus A is the *unique* element of $\mathcal{C}_{\psi_1}(\mathcal{U})$ that takes ψ_1 to η. Let $\mathcal{S}_{\psi_1,\ldots,\psi_n}$ be the family of all finite sums of the form

$$\sum_{i=0}^{n} A_i V_{\psi_1}^{\psi_i} \ .$$

This is the left module of $\mathcal{U}'$ generated by $\{I, V_{\psi_1}^{\psi_2}, \ldots, V_{\psi_1}^{\psi_n}\}$. It is the $\mathcal{U}'$-*linear span* of $\{I, V_{\psi_1}^{\psi_2}, \ldots, V_{\psi_1}^{\psi_n}\}$.

Let

$$\mathcal{M}_{\psi_1,\ldots,\psi_n} := (\mathcal{S}_{\psi_1,\ldots,\psi_n})\psi_1 \tag{13}$$

So

$$\mathcal{M}_{\psi_1,\ldots,\psi_n} \ = \ \left\{ \sum_{i=0}^{n} A_i\psi_i \ : \ A_i \in \mathcal{U}' \right\} \ .$$

We call this the *interpolation space* for $\mathcal{U}$ generated by $(\psi_1, \ldots, \psi_n)$. From the above discussion, it follows that for every vector $\eta \in \mathcal{M}_{\psi_1,\psi_2,\ldots,\psi_n}$ there exists a unique operator $A \in \mathcal{C}_{\psi_1}(\mathcal{U})$ such that $\eta = A\psi_1$, and moreover this A is an element of $\mathcal{S}_{\psi_1,\ldots,\psi_n}$.

3.3.1. Normalizing the Commutant. In certain essential cases (and we are not sure how general this type of case is) one can prove that an interpolation unitary V_ψ^η *normalizes* the commutant $\mathcal{U}'$ of the system in the sense that $V_\eta^\psi \mathcal{U}' V_\psi^\eta = \mathcal{U}'$. (Here, it is easily seen that $(V_\psi^\eta)^* = V_\eta^\psi$.) Write $V := V_\psi^\eta$. If V normalizes $\mathcal{U}'$, then the algebra, before norm closure, generated by $\mathcal{U}'$ and V is the set of all finite sums (trig polynomials) of the form $\sum A_n V^n$, with coefficients $A_n \in \mathcal{U}'$, $n \in \mathbb{Z}$. The closure in the strong operator topology is a von Neumann algebra. Now suppose further that *every power* of V is contained in $\mathcal{C}_\psi(\mathcal{U})$. This occurs only in special cases, yet it occurs frequently enough to yield some general methods. Then since $\mathcal{C}_\psi(\mathcal{U})$ is a SOT-closed linear subspace which is closed under left multiplication by $\mathcal{U}'$, this von Neumann algebra is contained in $\mathcal{C}_\psi(\mathcal{U})$, so its unitary group parameterizes a norm-path-connected subset of $\mathcal{W}(\mathcal{U})$ that contains ψ and η via the correspondence $U \to U\psi$.

In the special case of *wavelets*, this is the basis for the work that Dai and I did in [5, Chapter 5] on operator-theoretic interpolation of wavelets. In fact, we specialized there and *reserved* the term *operator-theoretic interpolation* to refer explicitly to the case when the interpolation unitaries normalize the commutant. In some subsequent work, we *loosened* this restriction yielding our more general definition given in this article, because there are cases of interest in which we weren't able to prove normalization. However, it turns out that if ψ and η are s-elementary wavelets (see section 4.4), then indeed V_ψ^η normalizes $\{D, T\}'$. (See Proposition 5.3.) Moreover, V_ψ^η has a very special form: after conjugating with the Fourier transform, it is a composition operator with a symbol σ that is a natural and very computable measure-preserving transformation of $\mathbb{R}$. In fact, it is precisely this special form for V_ψ^η that allows us to make the computation that it normalizes $\{D, T\}'$. On the other hand, we know of no pair (ψ, η) of wavelets for which V_ψ^η fails to normalize $\{D, T\}'$. The difficulty is simply that in general it is very hard to do the computations. This is stated as Problem 2 in the final section on *Open Problems*.

In the wavelet case $\mathcal{U}_{D,T}$, if $\psi \in \mathcal{W}(D, T)$ then it turns out that $\mathcal{C}_\psi(\mathcal{U}_{D,T})$ is in fact *much larger* than $(\mathcal{U}_{D,T})' = \{D, T\}'$, underscoring the fact that $\mathcal{U}_{D,T}$ is NOT a group. In particular, $\{D, T\}'$ is abelian while $\mathcal{C}_\psi(\mathcal{D}, \mathcal{T})$ is nonabelian for every wavelet ψ. (The proof of these facts are contained in [5].)

3.3.2. Interpolation Pairs of Wandering Vectors. In some cases where a pair ψ, η of vectors in $\mathcal{W}(\mathcal{U})$ are given it turns out that the unitary V in $\mathcal{C}_\psi(\mathcal{U})$ with $V\psi = \eta$ happens to be a *symmetry* (i.e. $V^2 = I$). Such pairs are called *interpolation pairs* of wandering vectors, and in the case where $\mathcal{U}$ is a wavelet system, they are called interpolation pairs of wavelets. Interpolation pairs are more prevalent in the theory, and in particular the wavelet theory, than one might expect. In this case (and in more complex generalizations of this) certain linear combinations of complete wandering vectors are themselves complete wandering vectors – not simply complete Riesz vectors.

Proposition 3.3. *Let $\mathcal{U}$ be a unitary system, let $\psi, \eta \in \mathcal{W}(\mathcal{U})$, and let V be the unique operator in $\mathcal{C}_\psi(\mathcal{U})$ with $V\psi = \eta$. Suppose*

$$V^2 = I.$$

Then

$$\cos\alpha \cdot \psi \; + i \sin\alpha \cdot \eta \in \mathcal{W}(\mathcal{U})$$

for all $0 \le \alpha \le 2\pi$.

The above result can be thought of as the *prototype* of our operator-theoretic interpolation results. It is the second most elementary case. (The most elementary case is described in the context of the exposition of Problem 4 in the final section.) More generally, the scalar α in Proposition 3.3 can be replaced with an appropriate *self-adjoint operator* in the commutant of $\mathcal{U}$. In the wavelet case, after conjugating with the Fourier transform, which is a unitary operator, this means that α can be replaced with a wide class of nonnegative dilation-periodic (see definition below) bounded measurable functions on $\mathbb{R}$.

4. Wavelet Sets

Wavelet sets belong to the theory of wavelets via the Fourier Transform. We will do most of this section in a tutorial-style, to make the concepts more accessible to students and colleagues who are not already familiar with them.

4.1. Fourier Transform

We will use the following form of the Fourier–Plancherel transform $\mathcal{F}$ on $\mathcal{H} = L^2(\mathbb{R})$, which is a form that is *normalized* so it is a unitary transformation, a property that is desirable for our treatment.

If $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$(\mathcal{F}f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)dt := \hat{f}(s), \tag{14}$$

and

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s)ds. \tag{15}$$

We have

$$(\mathcal{F}T_\alpha f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t-\alpha)dt = e^{-is\alpha}(\mathcal{F}f)(s).$$

So $\mathcal{F}T_\alpha\mathcal{F}^{-1}g = e^{-is\alpha}g$. For $A \in \mathcal{B}(\mathcal{H})$ let $\hat{A}$ denote $\mathcal{F}A\mathcal{F}^{-1}$. Thus

$$\widehat{T}_\alpha = M_{e^{-i\alpha s}}, \tag{16}$$

where for $h \in L^\infty$ we use M_h to denote the multiplication operator $f \to hf$. Since $\{M_{e^{-i\alpha s}}\colon\ \alpha \in \mathbb{R}\}$ generates the m.a.s.a. $\mathcal{D}(\mathbb{R}) := \{M_h\colon\ h \in L^\infty(\mathbb{R})\}$ as a von Neumann algebra, we have

$$\mathcal{F}\mathcal{A}_T\mathcal{F}^{-1} = \mathcal{D}(\mathbb{R}).$$

Similarly,

$$\begin{aligned}(\mathcal{F}D^n f)(s) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ist}(\sqrt{2})^n f(2^n t)dt \\ &= (\sqrt{2})^{-n}\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i2^{-n}st} f(t)dt \\ &= (\sqrt{2})^{-2}(\mathcal{F}f)(2^{2^{-n}s}) = (D^{-n}\mathcal{F}f)(s).\end{aligned}$$

So $\widehat{D}^n = D^{-n} = D^{*n}$. Therefore,

$$\widehat{D} = D^{-1} = D^*. \tag{17}$$

If f is an $L^2(\mathbb{R})$ function, as usual we write $\widehat{f}(s) = (\mathcal{F}(f))(s)$. If $\rho(s)$ is a real-valued function such that $\widehat{f}(s) = e^{i\rho(s)}|\widehat{f}(s)|$, we call $\rho(s)$ the *phase* of f. The phase is well defined *a.e.* modulo 2π-translation.

4.2. The Commutant of $\{D, T\}$

We have $\mathcal{F}\{D,T\}'\mathcal{F}^{-1} = \{\widehat{D},\widehat{T}\}'$. It turns out that $\{\widehat{D},\widehat{T}\}'$ has an elementary characterization in terms of Fourier multipliers:

Theorem 4.1.

$$\{\widehat{D},\widehat{T}\}' = \{M_h\colon\ h \in L^\infty(\mathbb{R}) \textit{ and } h(s) = h(2s) \textit{ a.e.}\}.$$

Proof. Since $\widehat{D} = D^*$ and D is unitary, it is clear that $M_h \in \{\widehat{D},\widehat{T}\}'$ if and only if M_h commutes with D. So let $g \in L^2(\mathbb{R})$ be arbitrary. Then (a.e.) we have

$$\begin{aligned}(M_h Dg)(s) &= h(s)(\sqrt{2}\ g(2s)), \quad \text{and} \\ (DM_h g)(s) &= D(h(s)g(s)) = \sqrt{h}(2s)g(2s).\end{aligned}$$

Since these must be equal a.e. for arbitrary g, we must have $h(s) = h(2s)$ a.e. □

Now let $E = [-2,-1)\cup[1,2)$, and for $n \in \mathbb{Z}$ let $E_n = \{2^n x\colon\ x \in E\}$. Observe that the sets E_n are disjoint and have union $\mathbb{R}\backslash\{0\}$. So if g is any uniformly bounded function on E, then g extends uniquely (a.e.) to a function $\tilde{g} \in L^\infty(\mathbb{R})$ satisfying

$$\tilde{g}(s) = \tilde{g}(2s), \qquad s \in \mathbb{R},$$

by setting

$$\tilde{g}(2^n s) = g(s), \qquad s \in E, n \in \mathbb{Z},$$

and $\tilde{g}(0) = 0$. We have $\|\tilde{g}\|_\infty = \|g\|_\infty$. Conversely, if h is any function satisfying $h(s) = h(2s)$ a.e., then h is uniquely (a.e.) determined by its restriction to E. This 1-1 mapping $g \to M_{\tilde{g}}$ from $L^\infty(E)$ onto $\{\widehat{D},\widehat{T}\}'$ is a $*$-isomorphism.

We will refer to a function h satisfying $h(s) = h(2s)$ a.e. as a 2-*dilation periodic function*. This gives a simple algorithm for computing a large class of wavelets from a given one, by simply modifying the *phase* (see also section 4.7):

$$\begin{aligned}&\textit{Given } \psi,, \textit{ let } \widehat{\psi} = \mathcal{F}(\psi), \textit{ choose a real-valued function } h \in L^\infty(E)\\ &\textit{arbitrarily, let } g = \exp(ih), \textit{ extend to a 2-dilation periodic}\\ &\textit{function } \tilde{g} \textit{ as above, and compute } \psi_{\tilde{g}} = \mathcal{F}^{-1}(\tilde{g}\widehat{\psi}).\end{aligned} \tag{18}$$

In the description above, the set E could clearly be replaced with $[-2\pi, -\pi) \cup [\pi, 2\pi)$, or with any other "dyadic" set $[-2a, a) \cup [a, 2a)$ for some $a > 0$.

4.3. The Shannon Wavelet

We now give an account of s-elementary and MSF-wavelets. The two most elementary dyadic orthonormal wavelets are the well-known *Haar wavelet* and *Shannon's wavelet* (also called the Littlewood–Paley wavelet). The Shannon set is the prototype of the class of wavelet sets.

Shannon's wavelet is the $L^2(\mathbb{R})$-function with Fourier transform $\widehat{\psi}_S = \frac{1}{\sqrt{2\pi}}\chi_{E_0}$ where

$$E_0 = [-2\pi, -\pi) \cup [\pi, 2\pi). \tag{19}$$

The argument that $\widehat{\psi}_S$ is a wavelet is in a way even more transparent than for the Haar wavelet. And it has the advantage of generalizing nicely. For a simple argument, start from the fact that the exponents

$$\{e^{i\ell s}\colon\ n \in \mathbb{Z}\}$$

restricted to $[0, 2\pi]$ and normalized by $\frac{1}{\sqrt{2\pi}}$ is an orthonormal basis for $L^2[0, 2\pi]$. Write $E_0 = E_- \cup E_+$ where $E_- = [-2\pi, -\pi)$, $E_+ = [\pi, 2\pi)$. Since $\{E_- + 2\pi, E_+\}$ is a partition of $[0, 2\pi)$ and since the exponentials $e^{i\ell s}$ are invariant under translation by 2π, it follows that

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}}\Big|_{E_0} \colon\ n \in \mathbb{Z} \right\} \tag{20}$$

is an orthonormal basis for $L^2(E_0)$. Since $\widehat{T} = M_{e^{-is}}$, this set can be written

$$\{\widehat{T}^\ell \widehat{\psi}_s \colon\ \ell \in \mathbb{Z}\}. \tag{21}$$

Next, note that any "dyadic interval" of the form $J = [b, 2b)$, for some $b > 0$ has the property that $\{2^n J\colon\ n \in \mathbb{Z}\}$, is a partition of $(0, \infty)$. Similarly, any set of the form

$$\mathcal{K} = [-2a, -a) \cup [b, 2b) \tag{22}$$

for $a, b > 0$, has the property that

$$\{2^n \mathcal{K}\colon\ n \in \mathbb{Z}\}$$

is a partition of $\mathbb{R}\backslash\{0\}$. It follows that the space $L^2(\mathcal{K})$, considered as a subspace of $L^2(\mathbb{R})$, is a complete wandering subspace for the dilation unitary $(Df)(s) = \sqrt{2}\ f(2s)$. For each $n \in \mathbb{Z}$,

$$D^n(L^2(\mathcal{K})) = L^2(2^{-n}\mathcal{K}). \tag{23}$$

So $\bigoplus_n D^n(L^2(\mathcal{K}))$ is a direct sum decomposition of $L^2(\mathbb{R})$. In particular E_0 has this property. So

$$D^n \left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}}\Big|_{E_0} :\ \ell \in \mathbb{Z} \right\} = \left\{ \frac{e^{2^n i\ell s}}{\sqrt{2\pi}}\Big|_{2^{-n}E_0} :\ \ell \in \mathbb{Z} \right\} \tag{24}$$

is an orthonormal basis for $L^2(2^{-n}E_0)$ for each n. It follows that

$$\{D^n \widehat{T}^\ell \widehat{\psi}_s :\ n, \ell \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Hence $\{D^n T^\ell \psi_s :\ n, \ell \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, as required.

For our work, in order to proceed with developing an operator-algebraic theory that had a chance of directly impacting concrete function-theoretic wavelet theory we needed a large supply of examples of wavelets which were elementary enough to work with. First, we found another "Shannon-type" wavelet in the literature. This was the Journe wavelet, which we found described on p. 136 in Daubechies book [8]. Its Fourier transform is $\widehat{\psi}_J = \frac{1}{\sqrt{2\pi}}\chi_{E_J}$, where

$$E_J = \left[-\frac{32\pi}{7}, -4\pi\right) \cup \left[-\pi, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi\right) \cup \left[4\pi, \frac{32\pi}{7}\right).$$

Then, thinking the old adage "where there's smoke there's fire!", we painstakingly worked out many more examples. So far, these are the basic building blocks in the *concrete* part of our theory. By this we mean the part of our theory that has had some type of direct impact on function-theoretic wavelet theory.

4.4. Definition of Wavelet Set

We define a *wavelet set* to be a measurable subset E of $\mathbb{R}$ for which $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of a wavelet. The wavelet $\widehat{\psi}_E := \frac{1}{\sqrt{2\pi}}\chi_E$ is called *s-elementary* in [5].

It turns out that this class of wavelets was also discovered and systematically explored completely independently, and in about the same time period, by Guido Weiss (Washington University), his colleague and former student E. Hernandez (U. Madrid), and his students X. Fang and X. Wang. Two of the papers of this group are [9] and [17], in which they are called MSF (minimally supported frequency) wavelets. In signal processing, the parameter s, which is the independent variable for $\widehat{\psi}$, is the *frequency* variable, and the variable t, which is the independent variable for ψ, is the *time* variable. It is not hard to show that no function with support a subset of a wavelet set E of strictly smaller measure can be the Fourier transform of a wavelet. (Here, the support of a measurable function is defined to be the set of points at which it does not vanish.) In other words, an MSF wavelet has *minimal* possible support in the frequency domain. However, the problem of whether the support set of any wavelet necessarily contains a wavelet set remains open. It was raised by this author (Larson) in a talk about ten years ago, and has been open for several years. We include it as Problem 3 in the final section of this article. A natural subproblem, which was posed in the same talk, asks whether a wavelet with minimal possible support in the frequency domain is in fact an MSF wavelet; or equivalently, is its support a wavelet set?

4.4.1. The Spectral Set Condition. From the argument above describing why Shannon's wavelet is, indeed, a wavelet, it is clear that *sufficient* conditions for E to be a wavelet set are

> (i) the normalized exponential $\frac{1}{\sqrt{2\pi}}e^{i\ell s}$, $\ell \in \mathbb{Z}$, when restricted to E should constitute an orthonormal basis for $L^2(E)$ (in other words E is a *spectral set* for the integer lattice $\mathbb{Z}$),

and

> (ii) The family $\{2^n E\colon\ n \in \mathbb{Z}\}$ of dilates of E by integral powers of 2 should constitute a measurable partition (i.e. a partition modulo null sets) of $\mathbb{R}$.

These conditions are also necessary. In fact if a set E satisfies (i), then for it to be a wavelet set it is obvious that (ii) must be satisfied. To show that (i) must be satisfied by a wavelet set E, consider the vectors

$$\widehat{D}^n \widehat{\psi}_E = \frac{1}{\sqrt{2\pi}}\chi_{2^{-n}E}, \qquad n \in \mathbb{Z}.$$

Since $\widehat{\psi}_E$ is a wavelet these must be orthogonal, and so the sets $\{2^n E\colon\ n \in\ \mathbb{Z}\}$ must be disjoint modulo null sets. It follows that $\{\frac{1}{\sqrt{2\pi}}e^{i\ell s}|_E\colon\ \ell \in \mathbb{Z}\}$ is not only an orthonormal set of vectors in $L^2(E)$, it must also *span* $L^2(E)$.

It is known from the theory of *spectral sets* (as an elementary special case) that a measurable set E satisfies (i) if and only if it is a generator of a measurable partition of $\mathbb{R}$ under translation by 2π (i.e. iff $\{E + 2\pi n\colon\ n \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{R}$). This result generalizes to spectral sets for the integral lattice in $\mathbb{R}^n$. For this elementary special case a direct proof is not hard.

4.5. Translation and Dilation Congruence

We say that measurable sets E, F are *translation congruent modulo* 2π if there is a measurable bijection $\phi\colon\ E \to F$ such that $\phi(s) - s$ is an integral multiple of 2π for each $s \in E$; or equivalently, if there is a measurable partition $\{E_n\colon\ n \in \mathbb{Z}\}$ of E such that

$$\{E_n + 2n\pi\colon\ n \in \mathbb{Z}\} \tag{25}$$

is a measurable partition of F. Analogously, define measurable sets G and H to be *dilation congruent modulo* 2 if there is a measurable bijection $\tau\colon\ G \to H$ such that for each $s \in G$ there is an integer n, depending on s, such that $\tau(s) = 2^n s$; or equivalently, if there is a measurable partition $\{G_n\}_{-\infty}^{\infty}$ of G such that

$$\{2^n G\}_{-\infty}^{\infty} \tag{26}$$

is a measurable partition of H. (Translation and dilation congruency modulo other positive numbers of course make sense as well.)

The following lemma is useful.

Lemma 4.2. *Let $f \in L^2(\mathbb{R})$, and let $E = \operatorname{supp}(f)$. Then f has the property that*

$$\{e^{ins} f\colon\ n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(E)$ if and only if

(i) *E is congruent to $[0, 2\pi)$ modulo 2π, and*
(ii) *$|f(s)| = \frac{1}{\sqrt{2\pi}}$ a.e. on E.*

If E is a measurable set which is 2π-translation congruent to $[0, 2\pi)$, then since

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}}\Big|_{[0,2\pi)}\colon\ \ell \in \mathbb{Z} \right\}$$

is an orthonormal basis for $L^2[0, 2\pi]$ and the exponentials $e^{i\ell s}$ are 2π-invariant, as in the case of Shannon's wavelet it follows that

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}}\Big|_{E}\colon\ \ell \in \mathbb{Z} \right\}$$

is an orthonormal basis for $L^2(E)$. Also, if E is 2π-translation congruent to $[0, 2\pi)$, then since

$$\{[0, 2\pi) + 2\pi n\colon\ n \in \mathbb{Z}\}$$

is a measurable partition of $\mathbb{R}$, so is

$$\{E + 2\pi n\colon\ n \in \mathbb{Z}\}.$$

These arguments can be reversed.

We say that a measurable subset $G \subset \mathbb{R}$ is a 2-*dilation generator* of a *partition* of $\mathbb{R}$ if the sets

$$2^n G := \{2^n s\colon\ s \in G\}, \qquad n \in \mathbb{Z} \tag{27}$$

are disjoint and $\mathbb{R} \backslash \cup_n 2^n G$ is a null set. Also, we say that $E \subset \mathbb{R}$ is a 2π-*translation generator of a partition* of $\mathbb{R}$ if the sets

$$E + 2n\pi := \{s + 2n\pi\colon\ s \in E\}, \qquad n \in \mathbb{Z}, \tag{28}$$

are disjoint and $\mathbb{R} \backslash \cup_n (E + 2n\pi)$ is a null set.

Lemma 4.3. *A measurable set $E \subseteq \mathbb{R}$ is a 2π-translation generator of a partition of $\mathbb{R}$ if and only if, modulo a null set, E is translation congruent to $[0, 2\pi)$ modulo 2π. Also, a measurable set $G \subseteq \mathbb{R}$ is a 2-dilation generator of a partition of $\mathbb{R}$ if and only if, modulo a null set, G is a dilation congruent modulo 2 to the set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.*

4.6. A Criterion

The following is a useful criterion for wavelet sets. It was published independently by Dai–Larson in [5] and by Fang–Wang (who were students of Guido Weiss) in [9] at about the same time (in December, 1994). In fact, it is amusing that the two papers had been submitted within two days of each other; only much later did we even learn of each other's work on wavelets and of this incredible timing.

Proposition 4.4. *Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a wavelet set if and only if E is both a 2-dilation generator of a partition (modulo null sets) of $\mathbb{R}$ and a 2π-translation generator of a partition (modulo null sets) of $\mathbb{R}$. Equivalently, E is a wavelet set if and only if E is both translation congruent to $[0, 2\pi)$ modulo 2π and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ modulo 2.*

Note that a set is 2π-translation congruent to $[0, 2\pi)$ iff it is 2π-translation congruent to $[-2\pi, \pi) \cup [\pi, 2\pi)$. So the last sentence of Proposition 4.4 can be stated: A measurable set E is a wavelet set if and only if it is both 2π-translation and 2-dilation congruent to the Littlewood–Paley set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.

4.7. Phases

If E is a wavelet set, and if $f(s)$ is any function with support E which has constant modulus $\frac{1}{\sqrt{2\pi}}$ on E, then $\mathcal{F}^{-1}(f)$ is a wavelet. Indeed, by Lemma 4.2 $\{\widehat{T}^{\ell} f\colon \ \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(E)$, and since the sets $2^n E$ partition $\mathbb{R}$, so $L^2(E)$ is a complete wandering subspace for $\widehat{D}$, it follows that $\{\widehat{D}^n \widehat{T}^{\ell} f\colon \ n, \ell \in \mathbb{Z}\}$ must be an orthonormal basis for $L^2(\mathbb{R})$, as required. In [9, 17] the term MSF-wavelet includes this type of wavelet. So MSF-wavelets can have arbitrary phase and s-elementary wavelets have phase 0. If ψ is a wavelet we say [5] that a real-valued function $\rho(s)$ is *attainable* as a phase of ψ if the function $e^{i\rho(s)}|\psi(s)|$ is also the Fourier transform of a wavelet. So *every* phase is *attainable* in this sense for an MSF or s-elementary wavelet. Attainable phases of wavelets have been studied in [5] and [26], in particular.

4.8. Some Examples of One-Dimensional Wavelet Sets

It is usually easy to determine, using the dilation-translation criteria, in Proposition 4.4, whether a given finite union of intervals is a wavelet set. In fact, to verify that a given "candidate" set E is a wavelet set, it is clear from the above discussion and criteria that it suffices to do two things.

1. Show, by appropriate partitioning, that E is 2-dilation-congruent to a set of the form $[-2a, -a) \cup [b, 2b)$ for some $a, b > 0$. and
2. Show, by appropriate partitioning, that E is 2π-translation-congruent to a set of the form $[c, c + 2\pi)$ for some real number c.

On the other hand, wavelet sets suitable for testing hypotheses can be quite difficult to construct. There are very few "recipes" for wavelet sets, as it were. Many families of such sets have been constructed for reasons including perspective, experimentation, testing hypotheses, etc., including perhaps the pure enjoyment of doing the computations – which are somewhat "puzzle-like" in nature. In working with the theory it is nice (and in fact we find it necessary) to have a large supply of wavelets on hand that permit relatively simple analysis.

For this reason we take the opportunity here to present for the reader a collection of such sets, mainly taken from [5], leaving most of the work in verifying that they are indeed wavelet sets to the reader.

We refer the reader to [6] for a proof of the existence of wavelet sets in $\mathbb{R}^{(n)}$, and a proof that there are sufficiently many to generate the Borel structure of

$\mathbb{R}^{(n)}$. These results are true for arbitrary expansive dilation factors. Some concrete examples in the plane were subsequently obtained by Soardi and Weiland, and others were obtained by Gu and by Speegle in their thesis work at A&M. Two had also been obtained by Dai for inclusion in the revised concluding remarks section of our Memoir [5].

In these examples we will usually write intervals as half-open intervals $[\cdot\,,\ )$ because it is easier to verify the translation and dilation congruency relations (1) and (2) above when wavelet sets are written thus, even though in actuality the relations need only hold modulo null sets.

(i) As mentioned above, an example due to Journe of a wavelet which admits no multiresolution analysis is the s-elementary wavelet with wavelet set

$$\left[-\frac{32\pi}{7}, -4\pi\right) \cup \left[-\pi, \frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi\right) \cup \left[4\pi, \frac{32\pi}{7}\right).$$

To see that this satisfies the criteria, label these intervals, in order, as J_1, J_2, J_3, J_4 and write $J = \cup J_i$. Then

$$J_1 \cup 4J_2 \cup 4J_3 \cup J_4 = \left[-\frac{32\pi}{7}, -\frac{16\pi}{7}\right) \cup \left[\frac{16\pi}{7}, \frac{32\pi}{7}\right).$$

This has the form $[-2a, a) \cup [b, 2b)$ so is a 2-dilation generator of a partition of $\mathbb{R}\backslash\{0\}$. Then also observe that

$$\{J_1 + 6\pi, J_2 + 2\pi, J_3, J_4 - 4\pi\}$$

is a partition of $[0, 2\pi)$.

(ii) The Shannon (or Littlewood–Paley) set can be generalized. For any $-\pi < \alpha < \pi$, the set

$$E_\alpha = [-2\pi + 2\alpha, -\pi + \alpha) \cup [\pi + \alpha, 2\pi + 2\alpha)$$

is a wavelet set. Indeed, it is clearly a 2-dilation generator of a partition of $\mathbb{R}\backslash\{0\}$, and to see that it satisfies the translation congruency criterion for $-\pi < \alpha \leq 0$ (the case $0 < \alpha < \pi$ is analogous) just observe that

$$\{[-2\pi + 2\alpha, 2\pi) + 4\pi, [-2\pi, -\pi + \alpha) + 2\pi, [\pi + \alpha, 2\pi + 2\alpha)\}$$

is a partition of $[0, 2\pi)$. It is clear that ψ_{E_α} is then a continuous (in $L^2(\mathbb{R})$-norm) path of s-elementary wavelets. Note that

$$\lim_{\alpha \to \pi} \widehat{\psi}_{E_\alpha} = \frac{1}{\sqrt{2\pi}} \chi_{[2\pi, 4\pi)}.$$

This is *not* the Fourier transform of a wavelet because the set $[2\pi, 4\pi)$ is not a 2-dilation generator of a partition of $\mathbb{R}\backslash\{0\}$. So

$$\lim_{\alpha \to \pi} \psi_{E_\alpha}$$

is not an orthogonal wavelet. (It is what is known as a Hardy wavelet because it generates an orthonormal basis for $H^2(\mathbb{R})$ under dilation and translation.) This example demonstrates that $\mathcal{W}(D, T)$ is *not* closed in $L^2(\mathbb{R})$.

(iii) Journe's example above can be extended to a path. For $-\frac{\pi}{7} \leq \beta \leq \frac{\pi}{7}$ the set

$$J_\beta = \left[-\frac{32\pi}{7}, -4\pi + 4\beta\right) \cup \left[-\pi + \beta, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi + \beta\right) \cup \left[4\pi + 4\beta, 4\pi + \frac{4\pi}{7}\right)$$

is a wavelet set. The same argument in (i) establishes dilation congruency. For translation, the argument in (i) shows congruency to $[4\beta, 2\pi + 4\beta)$ which is in turn congruent to $[0, 2\pi)$ as required. Observe that here, as opposed to in (ii) above, the limit of ψ_{J_β} as β approaches the boundary point $\frac{\pi}{7}$ *is* a wavelet. Its wavelet set is a union of 3 disjoint intervals.

(iv) Let $A \subseteq [\pi, \frac{3\pi}{2})$ be an arbitrary measurable subset. Then there is a wavelet set W, such that $W \cap [\pi, \frac{3\pi}{2}) = A$. For the construction, let

$$\begin{aligned} & B = [2\pi, 3\pi)\backslash 2A, \\ & C = \left[-\pi, -\frac{\pi}{2}\right) \backslash (A - 2\pi) \\ \text{and} \quad & D = 2A - 4\pi. \end{aligned}$$

Let

$$W = \left[\frac{3\pi}{2}, 2\pi\right) \cup A \cup B \cup C \cup D.$$

We have $W \cap [\pi, \frac{3\pi}{2}) = A$. Observe that the sets $[\frac{3\pi}{2}, 2\pi)$, A, B, C, D, are disjoint. Also observe that the sets

$$\left[\frac{3\pi}{2}, 2\pi\right), A, \frac{1}{2}B, 2C, D,$$

are disjoint and have union $[-2\pi, -\pi) \cup [\pi, 2\pi)$. In addition, observe that the sets

$$\left[\frac{3\pi}{2}, 2\pi\right), A, B - 2\pi, C + 2\pi, D + 2\pi,$$

are disjoint and have union $[0, 2\pi)$. Hence W is a wavelet set.

(v) Wavelet sets for arbitrary (not necessarily integral) dilation factors other then 2 exist. For instance, if $d \geq 2$ is arbitrary, let

$$\begin{aligned} A &= \left[-\frac{2d\pi}{d+1}, -\frac{2\pi}{d+1}\right), \\ B &= \left[\frac{2\pi}{d^2-1}, \frac{2\pi}{d+1}\right), \\ C &= \left[\frac{2d\pi}{d+1}, \frac{2d^2\pi}{d^2-1}\right) \end{aligned}$$

and let $G = A \cup B \cup C$. Then G is d-wavelet set. To see this, note that $\{A + 2\pi, B, C\}$ is a partition of an interval of length 2π. So G is 2π-translation-congruent to $[0, 2\pi)$.

Also, $\{A, B, d^{-1}C\}$ is a partition of the set $[-d\alpha, -\alpha) \cup [\beta, d\beta)$ for $\alpha = \frac{2\pi}{d^2-1}$, and $\beta = \frac{2\pi}{d^2-1}$, so from this form it follows that $\{d^nG\colon\ n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}\backslash\{0\}$. Hence if $\psi := \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_G)$, it follows that $\{d^{\frac{n}{2}}\psi(d^nt - \ell)\colon\ n, \ell \in \mathbb{Z}\}$ is orthonormal basis for $L^2(\mathbb{R})$, as required.

5. Operator-Theoretic Interpolation for Wavelet Sets

Operator-theoretic interpolation takes a particularly natural form for the special case of s-elementary (or MSF) wavelets that facilitates hands-on computational techniques in investigating its properties. Let E, F be a pair of wavelet sets. Then for (a.e.) $x \in E$ there is a unique $y \in F$ such that $x - y \in 2\pi\mathbb{Z}$. This is the *translation congruence* property of wavelet sets. Also, for (a.e.) $x \in E$ there is a unique $z \in F$ such that $\frac{x}{z}$ is an integral power of 2. This is the *dilation congruence* property of wavelet sets. (See section 2.5.6.)

There is a natural *closed-form algorithm* for the *interpolation unitary* $V_{\psi_E}^{\psi_F}$ which maps the wavelet basis for $\widehat{\psi}_E$ to the wavelet basis for $\widehat{\psi}_F$. Indeed, using both the translation and dilation congruence properties of $\{E, F\}$, one can explicitly compute a (unique) measure-preserving transformation $\sigma := \sigma_E^F$ mapping $\mathbb{R}$ onto $\mathbb{R}$ which has the property that $V_{\psi_E}^{\psi_F}$ is identical with the *composition operator* defined by:

$$f \mapsto f \circ \sigma^{-1}$$

for all $f \in L^2(\mathbb{R})$. With this formulation, compositions of the maps σ between different pairs of wavelet sets are not difficult to compute, and thus products of the corresponding interpolation unitaries can be computed in terms of them.

5.1. The Interpolation Map σ

Let E and F be arbitrary wavelet sets. Let $\sigma\colon\ E \to F$ be the 1-1, onto map implementing the 2π-translation congruence. Since E and F both generated partitions of $\mathbb{R}\backslash\{0\}$ under dilation by powers of 2, we may extend σ to a 1-1 map of $\mathbb{R}$ onto $\mathbb{R}$ by defining $\sigma(0) = 0$, and

$$\sigma(s) = 2^n\sigma(2^{-n}s) \quad \text{for} \quad s \in 2^nE, \quad n \in \mathbb{Z}. \tag{29}$$

We adopt the notation σ_E^F for this, and call it the *interpolation map* for the ordered pair (E, F).

Lemma 5.1. *In the above notation, σ_E^F is a measure-preserving transformation from $\mathbb{R}$ onto $\mathbb{R}$.*

Proof. Let $\sigma := \sigma_E^F$. Let $\Omega \subseteq \mathbb{R}$ be a measurable set. Let $\Omega_n = \Omega \cap 2^nE$, $n \ni \mathbb{Z}$, and let $E_n = 2^{-n}\Omega_n \subseteq E$. Then $\{\Omega_n\}$ is a partition of Ω, and we have $m(\sigma(E_n)) = m(E_n)$ because the restriction of σ to E is measure-preserving. So

$$\begin{aligned} m(\sigma(\Omega)) &= \sum_n m(\sigma(\Omega_n)) = \sum_n m(2^n\sigma(E_n)) \\ &= \sum_n 2^n m(\sigma(E_n)) = \sum_n 2^n m(E_n) \\ &= \sum_n m(2^n E_n) = \sum_n m(\Omega_n) = m(\Omega). \end{aligned}$$ □

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called 2-*homogeneous* if $f(2s) = 2f(s)$ for all $s \in \mathbb{R}$. Equivalently, f is 2-homogeneous iff $f(2^n s) = 2^n f(s)$, $s \in \mathbb{R}$, $n \in \mathbb{Z}$. Such a function is completely determined by its values on any subset of $\mathbb{R}$ which generates a partition of $\mathbb{R}\backslash\{0\}$ by 2-dilation. So σ_E^F is the (unique) 2-homogeneous extension of the 2π-transition congruence $E \to F$. The set of all 2-homogeneous measure-preserving transformations of $\mathbb{R}$ clearly forms a group under composition. Also, the composition of a 2-dilation-periodic function f with a 2-homogeneous function g is (in either order) 2-dilation periodic. We have $f(g(2s)) = f(2g(s)) = f(g(s))$ and $g(f(2s)) = g(f(s))$. These facts will be useful.

5.1.1. An Algorithm for the Interpolation Unitary. Now let

$$U_E^F := U_{\sigma_E^F}, \tag{30}$$

where if σ is any measure-preserving transformation of $\mathbb{R}$ then U_σ denotes the composition operator defined by $U_\sigma f = f \circ \sigma^{-1}$, $f \in L^2(\mathbb{R})$. Clearly $(\sigma_E^F)^{-1} = \sigma_F^E$ and $(U_E^F)^* = U_F^E$. We have $U_E^F \widehat{\psi}_E = \widehat{\psi}_F$ since $\sigma_E^F(E) = F$. That is,

$$U_E^F \widehat{\psi}_E = \widehat{\psi}_E \circ \sigma_F^E = \frac{1}{\sqrt{2\pi}}\chi_{_E} \circ \sigma_F^E = \frac{1}{\sqrt{2\pi}}\chi_{_F} = \widehat{\psi}_F.$$

Proposition 5.2. *Let E and F be arbitrary wavelet sets. Then $U_E^F \in \mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$. Hence $\mathcal{F}^{-1}U_E^F\mathcal{F}$ is the interpolation unitary for the ordered pair (ψ_E, ψ_F).*

Proof. Write $\sigma = \sigma_E^F$ and $U_\sigma = U_E^F$. We have $U_\sigma\widehat{\psi}_E = \widehat{\psi}_F$ since $\sigma(E) = F$. We must show

$$U_\sigma \widehat{D}^n \widehat{T}^l \widehat{\psi}_E = \widehat{D}^n \widehat{T}^l U_\sigma \widehat{\psi}_E, \quad n, l \in \mathbb{Z}.$$

We have

$$\begin{aligned} (U_\sigma \widehat{D}^n \widehat{T}^l \widehat{\psi}_E)(s) &= (U_\sigma \widehat{D}^n e^{-ils}\widehat{\psi}_E)(s) \\ &= U_\sigma 2^{-\frac{n}{2}} e^{-il2^{-n}s}\widehat{\psi}_E(2^{-n}s) \\ &= 2^{-\frac{n}{2}} e^{-il2^{-n}\sigma^{-1}(s)}\widehat{\psi}_E(2^{-n}\sigma^{-1}(s)) \\ &= 2^{-\frac{n}{2}} e^{-il\sigma^{-1}(2^{-n}s)}\widehat{\psi}_E(\sigma^{-1}(2^{-n}s)) \\ &= 2^{-\frac{n}{2}} e^{-il\sigma^{-1}(2^{-n}s)}\widehat{\psi}(2^{-n}s). \end{aligned}$$

This last term is nonzero iff $2^{-n}s \in F$, in which case $\sigma^{-1}(2^{-n}s) = \sigma_F^E(2^{-n}s) = 2^{-n}s + 2\pi k$ for some $k \in \mathbb{Z}$ since σ_F^E is a 2π-translation-congruence on F. It follows that $e^{-il\sigma^{-2}(2^{-n}s)} = e^{-il2^{-n}s}$. Hence we have

$$
\begin{aligned}
(U_\sigma \widehat{D}^n \widehat{T}^l \widehat{\psi}_E)(s) &= 2^{-\frac{n}{2}} e^{-ils^{-2n}s} \widehat{\psi}_F(2^{-n}s) \\
&= (\widehat{D}^n \widehat{T}^l \widehat{\psi}_F)(s) \\
&= (\widehat{D}^n \widehat{T}^l U_\sigma \widehat{\psi}_E)(s).
\end{aligned}
$$

We have shown $U_E^F \in \mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$. Since $U_E^F \widehat{\psi}_E = \widehat{\psi}_F$, the uniqueness part of Proposition 3.1 shows that $\mathcal{F}^{-1} U_E^F \mathcal{F}$ must be the interpolation unitary for (ψ_E, ψ_F). □

5.2. The Interpolation Unitary Normalizes the Commutant

Proposition 5.3. *Let E and F be arbitrary wavelet sets. Then the interpolation unitary for the ordered pair (ψ_E, ψ_F) normalizes $\{D, T\}'$.*

Proof. By Proposition 5.2 we may work with U_E^F in the Fourier transform domain. By Theorem 6, the generic element of $\{\widehat{D}, \widehat{T}\}'$ has the form M_h for some 2-dilation-periodic function $h \in L^\infty(\mathbb{R})$. Write $\sigma = \sigma_E^F$ and $U_\sigma = U_E^F$. Then

$$
U_\sigma^{-1} M_h U_\sigma = M_{h \circ \sigma^{-1}}. \tag{31}
$$

So since the composition of a 2-dilation-periodic function with a 2-homogeneous function is 2-dilation-periodic, the proof is complete. □

5.2.1. $\mathcal{C}_\psi(D,T)$ Is Nonabelian. It can also be shown ([5, Theorem 5.2 (iii)]) that if E, F are wavelet sets with $E \neq F$ then U_E^F is not contained in the double commutant $\{\widehat{D}, \widehat{T}\}''$. So since U_E^F and $\{\widehat{D}, \widehat{T}\}'$ are both contained in the local commutant of $\mathcal{U}_{\widehat{D},\widehat{T}}$ at $\widehat{\psi}_E$, this proves that $\mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$ is nonabelian. In fact (see [5, Proposition 1.8]) this can be used to show that $\mathcal{C}_\psi(D,T)$ is nonabelian for every wavelet ψ. We suspected this, but we could not prove it until we discovered the "right" way of doing the needed computation using s-elementary wavelets.

The above shows that a pair (E, F) of wavelets sets (or, rather, their corresponding s-elementary wavelets) admits operator-theoretic interpolation if and only if Group$\{U_E^F\}$ is contained in the local commutant $\mathcal{C}_{\widehat{\psi}_E}(\widehat{D}, \widehat{T})$, since the requirement that U_E^F normalizes $\{\widehat{D}, \widehat{T}\}'$ is automatically satisfied. It is easy to see that this is equivalent to the condition that for each $n \in \mathbb{Z}$, σ^n is a 2π-congruence of E in the sense that $(\sigma^n(s) - s)/2\pi \in \mathbb{Z}$ for all $s \in E$, which in turn implies that $\sigma^n(E)$ is a wavelet set for all n. Here $\sigma = \sigma_E^F$. This property hold trivially if σ is *involutive* (i.e. $\sigma^2 =$ identity).

5.2.2. The Coefficient Criterion. In cases where "torsion" is present, so $(\sigma_E^F)^k$ is the identity map for some finite integer k, the von Neumann algebra generated by $\{\widehat{D}, \widehat{T}\}'$ and $U := U_E^F$ has the simple form

$$
\left\{ \sum_{n=0}^{k} M_{h_n} U^n : \; h_n \in L^\infty(\mathbb{R}) \text{ with } h_n(2s) = h_n(s), \quad s \in \mathbb{R} \right\},
$$

and so each member of this "interpolated" family of wavelets has the form

$$\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{k} h_n(s)\chi_{\sigma^n(E)} \tag{32}$$

for 2-dilation periodic "coefficient" functions $\{h_n(s)\}$ which satisfy the necessary and sufficient condition that the operator

$$\sum_{n=0}^{k} M_{h_n}U^n \tag{33}$$

is unitary.

A standard computation shows that the map θ sending $\sum_0^k M_{h_n}U^n$ to the $k\times k$ function matrix (h_{ij}) given by

$$h_{ij} = h_{\alpha(i,j)}\circ\sigma^{-i+1} \tag{34}$$

where $\alpha(i,j) = (i+1)$ modulo k, is a $*$-isomorphism. This matricial algebra is the cross-product of $\{D,T\}'$ by the $*$-automorphism $ad(U_E^F)$ corresponding to conjugation with U_E^F. For instance, if $k=3$ then θ maps

$$M_{h_1} + M_{h_2}U_E^F + M_{h_3}(U_E^F)^2$$

to

$$\begin{pmatrix} h_1 & h_2 & h_3 \\ h_3\circ\sigma^{-1} & h_1\circ\sigma^{-1} & h_2\circ\sigma^{-1} \\ h_2\circ\sigma^{-2} & h_3\circ\sigma^{-2} & h_1\circ\sigma^{-2} \end{pmatrix}. \tag{35}$$

This shows that $\sum_0^k M_{h_n}U^n$ is a unitary operator iff the scalar matrix $(h_{ij})(s)$ is unitary for almost all $s\in\mathbb{R}$. Unitarity of this matrix-valued function is called the *Coefficient Criterion* in [5], and the functions h_i are called the interpolation coefficients. This leads to formulas for families of wavelets which are new to wavelet theory.

5.3. Interpolation Pairs of Wavelet Sets

For many interesting cases of note, the interpolation map σ_E^F will in fact be an *involution* of $\mathbb{R}$ (i.e. $\sigma\circ\sigma = id$, where $\sigma := \sigma_E^F$, and where id denotes the identity map). So torsion *will* be present, as in the above section, and it will be present in an essentially simple form. The corresponding interpolation unitary will be a *symmetry* in this case (i.e. a selfadjoint unitary operator with square I).

It is curious to note that verifying a simple operator equation $U^2 = I$ directly by matricial computation can be extremely difficult. It is much more computationally feasible to verify an equation such as this by pointwise (a.e.) verifying explicitly the relation $\sigma\circ\sigma = id$ for the interpolation map. In [5] we gave a number of examples of interpolation pairs of wavelet sets. We give below a collection of examples that has not been previously published: Every pair sets from the Journe family is an interpolation pair.

5.4. Journe Family Interpolation Pairs

Consider the parameterized path of *generalized Journe* wavelet sets given in Section 4.8 Item (iii). We have

$$J_\beta = \left[-\frac{32\pi}{7}, -4\pi - 4\beta\right) \cup \left[-\pi+\beta, -\frac{4\pi}{7}\right) \cup \left[\frac{4\pi}{7}, \pi+\beta\right) \cup \left[4\pi+4\beta, 4\pi+\frac{4\pi}{7}\right)$$

where the set of parameters β ranges $-\frac{\pi}{7} \le \beta \le \frac{\pi}{7}$.

Proposition 5.4. *Every pair* $(J_{\beta_1}, J_{\beta_2})$ *is an interpolation pair.*

Proof. Let $\beta_1, \beta_2 \in \left[-\frac{\pi}{7}, \frac{\pi}{7}\right)$ with $\beta_1 < \beta_2$. Write $\sigma = \sigma_{J_{\beta_2}}^{J_{\beta_1}}$. We need to show that

$$\sigma^2(x) = x \tag{*}$$

for all $x \in \mathbb{R}$. Since σ is 2-homogeneous, it suffices to verify (*) only for $x \in J_{\beta_1}$. For $x \in J_{\beta_1} \cap J_{\beta_2}$ we have $\sigma(x) = x$, hence $\sigma^2(x) = x$. So we only need to check (*) for $x \in (J_{\beta_1} \backslash J_{\beta_2})$. We have

$$J_{\beta_1} \backslash J_{\beta_2} = [-\pi+\beta_1, -\pi+\beta_2) \cup [4\pi+4\beta_1, 4\pi+4\beta_2).$$

It is useful to also write

$$J_{\beta_2} \backslash J_{\beta_1} = [-4\pi+4\beta_1, -4\pi+4\beta_2) \cup [\pi+\beta_1, \pi+\beta_2).$$

On $[-\pi+\beta_1, -\pi+\beta_2)$ we have $\sigma(x) = x+2\pi$, which lies in $[\pi+\beta_1, \pi+\beta_2)$. If we multiply this by 4, we obtain $4\sigma(x) \in [4\pi+4\beta_1, 4\pi+4\beta_2) \subset J_{\beta_1}$. And on $[4\pi+4\beta_1, 4\pi+4\beta_2)$ we clearly have $\sigma(x) = x - 8\pi$, which lies in $[-4\pi+4\beta_1, -4\pi+4\beta_2)$.

So for $x \in [-\pi+\beta_1, -\pi+\beta_2)$ we have

$$\sigma^2(x) = \sigma(\sigma(x)) = \frac{1}{4}\sigma(4\sigma(x)) = \frac{1}{4}[4\sigma(x) - 8\pi] = \sigma(x) - 2\pi = x + 2\pi - 2\pi = x.$$

On $[4\pi+4\beta_1, 4\pi+4\beta_2)$ we have $\sigma(x) = x - 8\pi$, which lies in $[-4\pi+4\beta_1, -4\pi+4\beta_2)$. So $\frac{1}{4}\sigma(x) \in [-\pi+\beta_1, -\pi+\beta_2)$. Hence

$$\sigma\left(\frac{1}{4}\sigma(x)\right) = \frac{1}{4}\sigma(x) + 2\pi$$

and thus

$$\sigma^2(x) = 4\sigma\left(\frac{1}{4}\sigma(x)\right) = 4\left[\frac{1}{4}\sigma(x) + 2\pi\right] = \sigma(x) + 8\pi = x - 8\pi + 8\pi = x$$

as required.

We have shown that for all $x \in J_{\beta_1}$ we have $\sigma^2(x) = x$. This proves that $(J_{\beta_1}, J_{\beta_2})$ is an interpolation pair.

□

6. Some Open Problems

We will discuss four problems on wavelets that we have investigated from an operator-theoretic point of view over the past ten years, together with some related problems. The set of orthonormal dyadic one-dimensional wavelets is a set of vectors in the unit sphere of a Hilbert space $H = L^2(\mathbb{R})$. It is natural to ask what are the topological properties of $\mathcal{W}(D,T)$ as a subset of the metric space H? This type of question is interesting from a pure mathematical point of view, speaking as an operator theorist, and it just may have some practical consequences depending on the nature and the degree of depth of solutions.

Problem 1: [Connectedness] This was the first global problem in wavelets that we considered from an operator-theoretic point of view. In [5] we posed a number of open problems in the context of the memoir. The first problem we discussed was Problem A (in [5]) which conjectured that $\mathcal{W}(D,T)$ is norm-arcwise-connected. It turned out that this conjecture was also formulated independently by Guido Weiss and his group (see [9], [17, [26]) from a harmonic analysis point of view (our point of view was purely functional analysis), and this problem (and related problems) was the primary stimulation for the creation of the WUTAM CONSORTIUM – a team of 14 researchers based at Washington University and Texas A&M University. (See [26], for the first publication of this group.) This *connectedness conjecture* was answered yes in [26] for the special case of the family of dyadic orthonormal MRA wavelets in $L^2(\mathbb{R})$, but still remains open for the family of *arbitrary* dyadic orthonormal wavelets in $L^2(\mathbb{R})$, as well as for the family of orthonormal wavelets for any fixed n and any dilation matrix in $\mathbb{R}^n$. A natural related problem which also remains open, is whether the set of Riesz wavelets is connected. An intermediate problem, which is also open, asks whether given two orthonormal wavelets is there a continuous path connecting them consisting of Riesz wavelets? (Some evidence for a positive answer to this problem is given by Proposition 6.1 below, which easily shows that every point on the convex path connecting two wavelets (i.e. $(1-t)\psi + t\eta)$is a Riesz wavelet *except* for perhaps the midpoint corresponding to $t = 0.5$. Thus this problem has an easy positive solution for many pairs of orthonormal wavelets, but no way has been found to get around the midpoint obstruction to show that all pairs are connected, perhaps by some exotic type of path.) A subproblem is the same problem but for the set of *frame-wavelets* $\mathcal{F}(D,T)$ (now widely called *framelets*). Is the set of all frame-wavelets connected?; or more specifically–is the set of all Parseval frame wavelets connected? The reader can easily deduce some *frame versions* of Proposition 6.1 using elementary spectral theory of operators which provides some quick-and-easy partial results on paths of frames. These are tantalizing, but the main problems still remain open.

All of these connectivity problems have counterparts for other unitary systems. For wavelet systems, they remain open (to our knowledge) for all dimensions n and all expansive matrix dilation factors. And for other systems, in particular in [14], we showed that for a fixed choice of modulation and translation parameters (necessarily, of course, with product ≤ 1) the set of Weyl-Heisenberg (or Gabor) frames is connected in this sense, and also it is norm-dense in $L^2(\mathbb{R})$ in the sense of Problem 2 below. Although the Weil-Heisenberg (aka Gabor) unitary systems are of a simpler operator-theoretic structure than the wavelet systems, and this permits the use of some techniques which do not work so well in the wavelet theory, even so this perhaps is another reason to think that general connectedness results are possible within the wavelet theory.

Problem 2: [Density] Is the set $\mathcal{RW}(D,T)$ of all Riesz wavelets *dense* in the norm topology in the Hilbert space $L^2(\mathbb{R})$? This was posed as a conjecture by Larson in a talk in August 1996 in a NATO conference held in Samos, Greece. It was posed in the same spirit as the connectivity problem above, in the sense that it asks about the topological nature of $\mathcal{RW}(D,T)$ as a subset of the metric space $L^2(\mathbb{R})$. Like the connectivity problem it is a *global* type of problem. A positive answer might be useful for applications if it could be given a some type of *quantitative interpretation.* A subproblem of this, which was discussed in several subsequent talks, is the same density problem but for the set of frame-wavelets $\mathcal{F}(D,T)$. (Of course a positive answer for wavelets would imply it for framelets.) Like the connectivity problem, this problem makes sense for the family of Riesz wavelets for any fixed n and any dilation matrix in $\mathbb{R}^n$.

One of the reasons for thinking that this conjecture may be positive is the following result, which we think is the most *elementary* application of operator-theoretic interpolation. It is abstracted from Chapter 1 of [5], although the form in [5] is a bit different.

Proposition 6.1. *Let $\mathcal{U}$ be a unitary system on a Hilbert space H. If ψ_1 and ψ_2 are in $\mathcal{W}(\mathcal{U})$, then*

$$\psi_1 + \lambda\psi_2 \in \mathcal{RW}(\mathcal{U})$$

for all complex scalars λ with $|\lambda| \neq 1$. More generally, if ψ_1 and ψ_2 are in $\mathcal{RW}(\mathcal{U})$ then there are positive constants $b > a > 0$ such that $\psi_1 + \lambda\psi_2 \in \mathcal{RW}(\mathcal{U})$ for all $\lambda \in \mathbb{C}$ with either $|\lambda| < a$ or with $|\lambda| > b$.

Proof. If $\psi_1, \psi_2 \in \mathcal{W}(\mathcal{U})$, let V be the unique unitary in $\mathcal{C}_{\psi_2}(\mathcal{U})$ given by Proposition 3.1 such that $V\psi_2 = \psi_1$. Then

$$\psi_1 + \lambda\psi_2 = (V + \lambda I)\psi_2.$$

Since V is unitary, $(V + \lambda I)$ is an invertible element of $\mathcal{C}_{\psi_2}(\mathcal{U})$ if $|\lambda| \neq 1$, so the first conclusion follows from Proposition 3.2. Now assume $\psi_1, \psi_2 \in \mathcal{RW}(\mathcal{U})$. Let A be the unique invertible element of $\mathcal{C}_{\psi_2}(\mathcal{U})$ such that $A\psi_2 = \psi_1$, and write $\psi_1 + \lambda\psi_2 = (A + \lambda I)\psi_2$. Since A is bounded and invertible there are $b > a > 0$ such that

$$\sigma(A) \subseteq \{z \in \mathbb{C} : a < |z| < b\}$$

where $\sigma(A)$ denotes the spectrum of A, and the same argument applies.

□

The above proposition indicates that Riesz wavelets are plentiful. As mentioned above, by writing $(1-t)\psi_2 + t\psi_1 = ((1-t)V + tI)\psi_1$, and using the fact that the local commutant is a linear space so contains $(1-t)V + tI$, it follows that a convex combination of orthonormal wavelets is a Riesz wavelet except possibly for the mid-point corresponding to $t = 0.5$. So, if ψ and η are orthonormal wavelets, the line in the vector space $L^2(\mathbb{R})$ containing the pair ψ, η is in the norm closure of the set of Riesz wavelets. Operator theoretic interpolation shows that more general linear combinations of finite families of Riesz wavelets are very often Riesz wavelets. And if one considers a finite family of wavelet sets $\mathcal{C}$, with union $\mathcal{S}$, then the restricted sets of Riesz wavelets, or orthonormal wavelets, or Parseval framelets, or framelets, which are restricted in the sense that they have their frequency support contained in $\mathcal{S}$, is always connected if the family of wavelet sets is an *interpolation family*, and these restricted sets of wavelets are very often connected even if the interpolation family criterion fails for $\mathcal{C}$. (In fact, it is a conjecture that these restricted sets of wavelets are *always* connected.) Moreover, these restricted sets are *dense* in $L^2(S)$ (considered as a subspace of $L^2(\mathbb{R})$) if the family $\mathcal{C}$ is an interpolation family, and it is yet another conjecture that they are *always* dense in $L^2(S)$. It is also known (see [5] for instance) that the linear span $\mathcal{W}(D,T)$ is dense in $L^2(\mathbb{R})$. So these facts together suggest that it is probably true that the set of Riesz wavelets $\mathcal{RW}(D,T)$ is dense in $L^2(\mathbb{R})$. However, the problem remains open. Also, as mentioned above under the *connectivity* problem, one can easily deduce some *frame versions* of Proposition 6.1 using elementary spectral theory of operators which provides some quick-and-easy partial results on density of Riesz wavelets and framelets. However, the general problem for Riesz wavelets remains open. (For the density problem for framelets, we mention that Marcin Bownick has recently obtained a significant positive result! It appears that he has solved the problem positively for *framelets*. But apparently it remains open for Riesz wavelets.)

As with the connectedness problem, all of these problems have counterparts for other unitary systems. For wavelet systems, they remain open (to our knowledge) for all dimensions n and all expansive matrix dilation factors (except for the recent interesting framelet density result of Bownick mentioned above). And for other systems, as mentioned in the context of Problem 1, in [14] we showed that for a fixed choice of modulation and translation parameters with product ≤ 1 the set of Weyl-Heisenberg (or Gabor) frames is dense in $L^2(\mathbb{R})$ in this sense. This is another reason to think that general density results might be possible within the wavelet theory.

Problem 3: [Frequency Support] (See section 4.4.) [Must the support of the Fourier transform of a wavelet contain a wavelet set?] This conjecture was posed about 10 years ago, by Larson, and the problem still remains open for the case of dimension 1 and dilation factor 2. It makes sense for any finite dimension n and any matrix dilation, and it apparently remains unsolved in any case. It has been studied by several researchers, and Z. Rzeszotnik, in particular, has made some progress on the problem. A related problem, also posed by Larson, (see section 4.4) asks whether a wavelet which has minimal support in the frequency domain is necessarily an MSF wavelet. (In other words: Is a minimal support set in the frequency domain necessarily a wavelet set).

Problem 4: [Normalization] (See section 3.3.1.) If $\{\psi, \eta\}$ is a pair of dyadic orthonormal wavelets, does the interpolation unitary V_ψ^η normalize $\{D, T\}'$? As mentioned above, the answer is yes if ψ and η are s-elementary wavelets. This problem makes sense for orthonormal wavelets in higher dimensions for matrix dilation factors, and for other scalar dilations in one dimension. We know of no counterexample for any of these cases. However, the problem might just lie in the fact that in most cases, other than for wavelet sets, we have no reasonable techniques for doing the computations. This problem was also discussed in context in Section 3.3.1, and it could be the most important problem remaining in the direction of further development of the unitary system approach to wavelet theory.

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Clifford Analysis and the Continuous Spherical Wavelet Transform

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Abstract. We present a group-theoretical approach for the continuous wavelet transform on the sphere S^{n-1}, based on the Lorentz group $\mathrm{Spin}(1,n)$ (the conformal group of the unit sphere). We introduce transformations on the sphere based on the decomposition of the group $\mathrm{Spin}(1,n)$ into the maximal compact subgroup of rotations ($\mathrm{Spin}(n)$) and the set of Möbius transformations in $\mathbb{R}^n$ of the form $\varphi_a(x) = (x-a)(1+ax)^{-1}, |a| < 1$. This approach presents an advantage of allowing the full use of the whole of the conformal group $\mathrm{Spin}(1,n)$, and in such way, it is a generalization of the continuous wavelet transform defined by J. P. Antoine and P. Vandergheynst (see [1], [2]). We will give an account of the influence of the parameter a arising in the definition of dilatations / contractions on the sphere. Finally we give different representations (with different properties) for the Hilbert space $L_2(S^{n-1})$ and the Hardy space H^2.

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1. Introduction

Wavelet analysis as proven useful in a myriad of applications due to the ability of wavelets to resolve localized signal content in both scale and space (see [3]). Many of these applications, however, are restricted to data defined in Euclidean space: the 1-dimensional line (signal processing), the 2-dimensional plane (image processing) and, occasionally, higher dimensions. Nevertheless, data are often measured or

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defined on other manifolds, such as the sphere. A number of attempts have been made to extend wavelets to the sphere, mainly via stereographic projection. Those are not really satisfactory, due to the disregard of the spherical geometry and the difficulty of defining dilatations on the sphere. Nevertheless it is possible to introduce local dilatations on the sphere if one uses the conformal group, that is, the Lorentz group $SO(1,n)$. In [1] the authors use, for the 2-sphere case, the Iwasawa decomposition of $SO_0(1,3)$ (or KAN decomposition, where K is the maximal compact subgroup, $A = SO_0(1,1) \cong \mathbb{R} \cong \mathbb{R}_*^+$ is the subgroup of Lorentz boosts in the $z-$direction and $N \cong \mathbb{C}$ is a two dimensional abelian subgroup). They use the parameter space $X \cong SO_0(1,3)/N \cong SO(3) \cdot \mathbb{R}_*^+$, i.e., the product of $SO(3)$ for motions and $\mathbb{R}_*^+$ for dilatations on S^2. We want to remark that when we impose this parameter space then some information is lost, since we do not make full use of the whole conformal group.

Here we want to define a continuous wavelet transform on the unit sphere which makes full use of the whole conformal group of the sphere. For that purpose we use some well known facts in Clifford analysis, where the study of the invariance group of null solutions of the Euclidean Dirac operator is of major importance (see [4], [5]). In the case of the sphere this group coincides with the group of Möbius transformations leaving the unit ball invariant [6]. One possible description of this group is in terms of a projective identification of the points in the Euclidean space $\mathbb{R}^n$ with the rays in the null cone in $\mathbb{R}^{1,n+1}$ (see [7], [8], [9]). Hence, the Möbius group is identified with the group $\mathrm{Spin}(1, n+1)$. This identification has been the main theme of several works on Clifford analysis (see [8], [10]). Also related with this approach is the study of the Clifford Analysis on hyperbolic spaces, and this due to the fact that the subgroup $\mathrm{Spin}(1,n)$ of Möbius transformations leaving the unit sphere invariant is the isometry group of these non-Euclidean geometries. For an overview of the function theory in the hyperbolic unit ball we refer the work of D. Eelbode ([10]). Furthermore, for the connection between wavelet theory and Clifford analysis we also would like to refer to [11], [12], [13] and [14].

2. Preliminaries

Let $\mathbb{R}^{p,q}$ denote the $n-$dimensional vectorial space over $\mathbb{R}$ ($n = p+q$) endowed with an orthonormal basis $e_i, i = 1, ..., n$, and with signature (p,q) induced by the non-degenerate bilinear form $B(x,y)$ such that $B(e_i,e_i) = -1$ for $1 \leq i \leq p$ and $B(e_i,e_i) = 1$ for $p < i \leq n$. We define $\mathbb{R}_{p,q}$ as the universal real algebra generated by $\mathbb{R}^{p,q}$ which preserves the bilinear form $B(x,y)$. Hence we have $e_i^2 = -B(e_i,e_i), i = 1, \ldots, n$ and $e_ie_j + e_je_i = 0, i \neq j$. For a vector x we have that $x^2 = -B(x,x)$ is real valued. A vector is said to be invertible if and only if it is non-isotropic. In $\mathbb{R}^{0,n}$ we have that each non-zero vector y is invertible

We define the Clifford conjugation $a \mapsto \overline{a}$ by $\overline{ab} = \overline{b}\overline{a}$, $\overline{e_i} = -e_i$, and $\overline{1} = 1$. As a consequence, the inverse of a vector y is given by $y^{-1} = \overline{y}/|y|^2$. We remark that due to the non-commutative character of Clifford algebras, the inverse at left is in general different from the inverse at right. Usually we denote by $\frac{x}{y}$ the product xy^{-1}, there is, by means of the right-hand side inverse. The particular linear combination of basic elements $e_{i_1} \dots e_{i_k}$, $(1 \leq i_1 < \dots < i_k \leq n)$, with equal length k is designated a $k-$vector and we shall denote by $[x]_k$ the k-vector part of $x \in \mathbb{R}_{p,q}$. The linear subspace over $\mathbb{R}$ spanned by the elements of equal length k is to be called $\mathbb{R}^k_{p,q}$ the space of $k-$vectors.

We introduce the Spin group $\mathrm{Spin}(p, q)$ of all even finite products of invertible vectors s such that $s\overline{s} = \pm 1$. For each $s \in \mathrm{Spin}(p, q)$ we have that the mapping $\chi(s) : x \mapsto \chi(s)x = sxs'^{-1}$ is a special orthogonal transformation, thus setting $\mathrm{Spin}(p, q)$ as a double covering of $SO(p, q)$.

3. Conformal Group of the Unit Sphere

Let us now take a look into the special case of the conformal group over the sphere. We can parameterize this group in the form $M(B^n) \sim SO(n) \times B^n$ where $SO(n)$ is the maximal compact subgroup of $M(B^n)$ and B^n is isomorphic to a group of Möbius transformations acting on the unit ball (see [8]). We consider the set of Möbius transformations

$$\varphi_a(x) = (x - a)(1 + ax)^{-1}, \quad |a| < 1 \tag{3.1}$$

which maps the unit ball onto itself and also the sphere onto the sphere.

The composition of two such Möbius transformations is again (up to a rotation) a Möbius transformation $\varphi_a \circ \varphi_b(x) = q\varphi_{(1-ab)^{-1}(a+b)}(x)\overline{q}$, where $q = \frac{1-ab}{|1-ab|}$. We denote by $a \times b = (1 - ab)^{-1}(a + b)$ the symbol of the new Möbius transformation. The symbol satisfies the relation $(1 - ab)^{-1}(a + b) = (a + b)(1 - ba)^{-1}$. We notice that the neutral element under this group operation is $\varphi_0 \equiv Id$ while for an inversion we have $\varphi_a^{-1}(x) = \varphi_{-a}(x)$. We have that $G = (M(B^n), \circ)$ is a (non-abelian) locally compact group. There exists a natural isomorphism between this group and the group of points $G^* = (B^n, \times)$ by means of an identification of each $\varphi_a \leftrightarrow a \in B^n$ and $\varphi_a \circ \varphi_b \leftrightarrow a \times b$.

Of special importance for this paper are the following two types of subgroups.

Example. **Subgroups of dimension $n-1$:** let $\omega \in S^{n-1}$. We consider the hyperplane $< \omega, x >= 0$ and we define the ball B^{n-1} as the intersection of the unit ball with this hyperplane. Then we have:

Proposition 3.1. *(see [15]) The set of points $a \in B^{n-1}$ together with the group operation $\times$ forms a subgroup of G^*.*

Example. **Subgroups of dimension one:** let L be the segment resulting from the intersection of the unit ball with the straight line passing by the origin and spanned by ω. Then we have:

Proposition 3.2. *(see [15]) The set of points $a \in L$ together with the group operation $\times$ forms a subgroup of G^* of dimension one.*

We remark that for each $a \in B^n$ the points $a/|a|$ and $-a/|a|$ are the fixed points of φ_a.

4. Hyperbolic Model

For the construction of a theory of wavelets the study of dilatations is of foremost importance. In the case of the sphere these dilatations are not given by simple Euclidean dilatations through inverse stereographic projection, but by hyperbolic rotations. In what follows we consider the Clifford Algebra $\mathbb{R}_{1,n}$, together with the special identification $\epsilon := e_{n+1}$, the vector that spans the time-axis.

A pure boost is viewed as a transformation $\mathcal{B}(\omega)$ which belongs to the Lie algebra generated by the bi-vectors of the form $\epsilon\omega$, with $\omega \in S^{n-1}$. It has the general form

$$s = \cosh\frac{\alpha}{2} + \epsilon\omega\sinh\frac{\alpha}{2}, \alpha \in \mathbb{R}, \omega \in S^{n-1} \tag{4.1}$$

and it acts on space-time vectors according to the transformation rules $X \to Y = sX\overline{s}$, and on functions via the (Spin-invariant) $L-$representation $F(X) \to L(s)F(X) = sF(\overline{s}Xs)$, or $H-$representation $F(X) \to H(s)F(X) = sF(\overline{s}Xs)\overline{s}$.

Proposition 4.1. *Let $\xi = \sum_{i=1}^n \xi_i e_i$ be a point on the sphere and s be of the form (4.1).*

Then the boost $s\xi\overline{s}$ yields the point on the sphere

$$\xi' = \sum_{i=1}^{n} \frac{\xi_i + ((\cosh\alpha - 1) < \xi, \omega > - \sinh\alpha)\omega_i}{\cosh\alpha - \sinh\alpha < \xi, \omega >} e_i \tag{4.2}$$

We can relate transformations (3.1) and (4.2) in the following way:

Proposition 4.2 (see [8]). *We assume, in (3.1), $a = t\omega$, with $-1 < t < 1$, and $\omega \in S^{n-1}$. Then transformations (3.1) and (4.2) are related by means of*

$$\alpha = \ln\left(\frac{1+t}{1-t}\right) \quad \text{and} \quad t = \frac{e^\alpha - 1}{e^\alpha + 1} = \tanh\left(\frac{\alpha}{2}\right).$$

As a result we obtain an isomorphism between the subgroup of Lorentz boosts in a fixed direction $\omega \in S^{n-1}$ and the subgroup of Möbius transformations of dimension one mentioned in proposition 3.2. Moreover, a pure boost $\mathcal{B}(\omega)$ can always be described as the composition $R(e_n, \omega) \circ \mathcal{B}(e_n) \circ R(\omega, e_n)$, where $R(\omega, \xi)$ stands for the rotation mapping $\omega \in S^{n-1}$ into $\xi \in S^{n-1}$. Therefore it is sufficient to consider pure boosts in the e_n-direction. We will consider the subgroup $\mathrm{Spin}(1,1)$ as the subgroup of Lorentz boosts in the e_n-direction. Its action on a given point $\omega = \{\theta_j, \phi\}_{j=1}^{n-2}$ of S^{n-1} is fully determined by

$$\omega \mapsto \omega_\alpha = \{(\theta_j)_\alpha, \phi_\alpha\}_{j=1}^{n-2}, \tag{4.3}$$

where

$$(\theta_j)_\alpha = \theta_j \quad \text{and} \quad \tan\frac{\phi_\alpha}{2} = \mathrm{e}^\alpha \tan\frac{\phi}{2}. \tag{4.4}$$

This action corresponds to a pure dilatation on the sphere and it is exactly the usual Euclidean dilatation lifted on S^{n-1} by inverse stereographic projection (see [2]). We will show in the next section that the local dilatation around the North Pole depends on two parameters (not only one as in [2]) if we use the whole conformal group of the sphere.

It is well known that the group $\mathrm{SO}(1,n)$ has two different decompositions, the so-called Iwasawa decomposition (or KAN-decomposition) and the Cartan decomposition (or KAK-decomposition) (see [16] and [17]). We now show how to obtain the KAK-decomposition for the $\mathrm{Spin}(1,n)$ group. We consider the following elements of $\mathrm{Spin}(n)$

$$\begin{aligned} s_i &= \cos\frac{\theta_i}{2} + e_1 e_{i+1} \sin\frac{\theta_i}{2}, \\ s_{n-1} &= \cos\frac{\phi}{2} + e_n e_1 \sin\frac{\phi}{2}, \end{aligned}$$

with $0 \leq \theta_i < 2\pi, i = 1, \ldots, n-2$, and $0 \leq \phi \leq \pi$. We identify the element $s = s_1 \ldots s_{n-2} s_{n-1}$ with the element $\xi(\theta_1, \ldots, \theta_{n-2}, \phi) \in S^{n-1} = \mathrm{Spin}(n)/\mathrm{Spin}(n-1)$. Then we obtain the following polar decomposition.

Lemma 4.3. *For s like above we have $\varphi_a(x) = \varphi_{sre_n\overline{s}}(x) = s\varphi_{re_n}(\overline{s}xs)\overline{s}$, where $r = |a| \in [0, 1[$.*

Thus, a Möbius transformation can be described in terms of a point $a \in x_n^+ - \text{axis} \cap B^n$ and a convenient rotation induced by s. If we apply to the right-hand side of this identity the rotation present in the usual $\mathrm{Spin}(1,n)$ decomposition (see [8]) we derive the KAK decomposition for an arbitrary element of the group $\mathrm{Spin}(1,n)$.

5. Influence of the Parameter a on Spherical Calottes

In this section we describe the influence of the parameter $a \in B^n$ on the generation of the new spherical calotte obtained by the application of a Möbius transformation φ_a to a given calotte. Without loss of generality we consider a spherical calotte $\mathcal{U}_h = \{x \in S^{n-1} : x_n \geq h\}$ centered at the North Pole, given in polar coordinates by

$$\left\{ \begin{array}{rcl} x_1 &=& \cos(\theta_1')\cos(\theta_2')\cdots\cos(\theta_{n-2}')\sin(\phi') \\ x_2 &=& \sin(\theta_1')\cos(\theta_2')\cdots\cos(\theta_{n-2}')\sin(\phi') \\ x_3 &=& \sin(\theta_2')\cos(\theta_3')\cdots\cos(\theta_{n-2}')\sin(\phi') \\ \vdots && \\ x_{n-1} &=& \sin(\theta_{n-2}')\sin(\phi') \\ x_n &=& \cos(\phi') \end{array} \right.$$

with $\theta_1' \in [0, 2\pi[, \theta_i' \in [0, \pi[, i \in \{2, \ldots, n-2\}$ and $\phi' \in [0, \phi_0]$, for a fixed $\phi_0 \in [0, \pi]$, with $h = \cos(\phi_0)$.

Consider now the sphere S^{n-2} in the hyperplane $x_n = h$

$$\begin{cases} x_1^2 + x_2^2 + \ldots + x_{n-1}^2 &= 1 - h^2 \\ x_n &= h \end{cases}.$$

We will consider this sphere as the *support of the spherical calotte* $\mathcal{U}_h$. Obviously $\varphi_a(S^{n-2})$ is a new sphere (say, S_*^{n-2}) and it stands for the support of the new spherical calotte. For a complete description of the new spherical calottes obtained see [15].

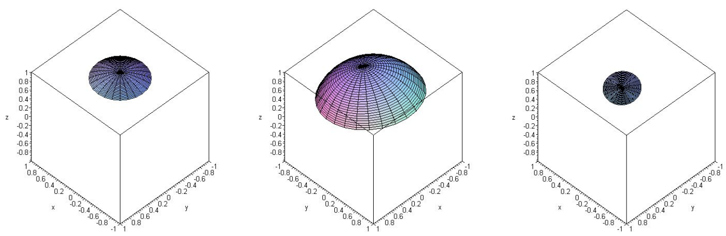
Definition 5.1. The image of the North Pole under the action of φ_a will be called **attractor point** and it will be denoted by **A**. It is given by:

$$A = \begin{bmatrix} \frac{2a_1(a_n-1)}{1+|a|^2-2a_n} \\ \\ \vdots \\ \frac{2a_{n-1}(a_n-1)}{1+|a|^2-2a_n} \\ \frac{(1-|a|^2+2a_n(a_n-1)}{1+|a|^2-2a_n} \end{bmatrix}. \tag{5.1}$$

Given an initial spherical calotte $\mathcal{U}_h$, its image $\varphi_a(\mathcal{U}_h)$ is a new spherical calotte, say $\mathcal{U}_{h,a}$, centered in general at a different point on the unit sphere, whose distance to its support hyperplane is given by

$$dist = \frac{2r\cos\phi - h(1+r^2)}{\sqrt{k}} + 1 \tag{5.2}$$

where $a \equiv a(r, \theta_1, \theta_2, \ldots, \theta_{n-2}, \phi)$, $k = 4r^2(r\cos\phi - h)^2 \sin^2\phi + (1 - r^2 + 2r\cos\phi$ $(r\cos\phi - h))^2$, that is, this distance only depends on the parameters r and ϕ. Then we obtain a local analysis on the sphere given by the family of neighborhoods $\{\mathcal{U}_{h,r,\phi}^{\theta_1,\cdots,\theta_{n-2}} : r \in [0,1[, \theta_1 \in [0,2\pi[, \theta_2, \ldots, \theta_{n-2}, \phi \in [0,\pi]$. These neighborhoods correspond to dilatations or contractions of the original calotte $\mathcal{U}_h$ under consideration. We illustrate this with an example in $\mathbb{R}^3$:



1 2 3

FIGURE 1. Spherical calotte for $h = \cos(\pi/6) = \sqrt{3}/2$: **1** - $\mathcal{U}_h$, **2** - $r = 1/2, \theta = 5\pi/3, \phi = \pi/6$, **3** - $r = 3/10, \theta = 5\pi/3, \phi = 7\pi/9$.

We have the advantage of having a preferable contraction inside the calotte given by the position of the *attractor point*. The parameters $\theta_1, \ldots, \theta_{n-2}$ contribute to the localization of the attractor point.

It is possible to define for each fixed $h \in [-1,1]$ two different regions on the unit ball that will be called *dilatation* and *contraction regions*, respectively. These two regions are separated by a surface $\mathcal{S}$ generated by a revolution of the arc $\overrightarrow{\gamma}(r) = (r(1-(hr)^2)^{1/2}, 0, \ldots, 0, r^2 h), r \in [0,1[$ in turn of the x_n-axis. The dilatation region corresponds to the region in the unit ball above the surface $\mathcal{S}$ while the contraction region corresponds to the region below the surface $\mathcal{S}$. We remark that $\varphi_a(\mathcal{U}_h), a \in \mathcal{S}$, have all equal area. They only differ by the localization of the attractor point.

The particular case of [1] and [2] is obtained assuming the values $\phi = 0$ (x_n^+-axis $\cap B^n$ - dilatation region) and $\phi = \pi$ (x_n^--axis $\cap B^n$ - contraction region). These two half axis can be used to generate/construct a sequence of approximation spaces. However it is possible to choose other domains for the parameter's variation and thus we would obtain different sequences of approximation spaces.

6. The Continuous Spherical Wavelet Transform

We will consider two different Hilbert spaces. The first is the usual space of square integrable functions on the sphere, namely, the space $L_2(S^{n-1})$ and the second is the monogenic Hardy space $H^2 \subset L_2(S^{n-1})$, which means the functions which can be considered as boundary values of monogenic functions on the unit ball.

On the space $L_2(S^{n-1})$ we use the following inner product and norm

$$< f, g >= \int_{S^{n-1}} \overline{f(x)} g(x) dS(x), \tag{6.1}$$

and

$$||f||^2 = 2^n \int_{S^{n-1}} [\overline{f(x)} f(x)]_0 dS(x), \tag{6.2}$$

where $[\lambda]_0$ denotes the real part of the Clifford number λ and $dS(x)$ is the normalized $Spin(n)-$invariant measure on S^{n-1}. We consider the following unitary operators

$$R_1(s)f(x) = f(\overline{s}xs), \;\; s \in \mathrm{Spin}(n) \tag{6.3}$$

and

$$D_1(a)f(x) = \left(\frac{1-|a|^2}{|1-ax|^2} \right)^{\frac{n-1}{2}} f(\varphi_{-a}(x)), \, a \in B^n. \tag{6.4}$$

We remark that in the case of $a = re_n$, with $r \in]-1,1[$, via the change of variables $r = \frac{u-1}{u+1}$, $u > 0$, we obtain the operator

$$D_1(u)f(x) = \lambda_1(u,\phi) f(\omega_{1/u}), \tag{6.5}$$

where

$$\lambda_1(u,\phi) = \left(\frac{4u^2}{((u^2-1)\cos\phi + (u^2+1))^2} \right)^{\frac{n-1}{4}}$$

and $\omega_{1/u}$ is the notation used in (4.3) with $\alpha = \ln(1/u)$. This operator is the same operator used in [2].

Based on the two operators defined above we consider the representation

$$\begin{aligned}\pi_1(s,a)f(x) &= R_1(s)\circ D_1(a)f(x) = \\ &= \left(\frac{1-|a|^2}{|1-a\overline{s}xs|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-a}(\overline{s}xs)) \qquad (6.6)\end{aligned}$$

which can be proved to be equivalent to the representation

$$\tilde{\pi}_1(s,b) = \left(\frac{1-|b|^2}{|1-bx|^2}\right)^{\frac{n-1}{2}} f(\overline{s}\varphi_{-b}(x)s) \qquad (6.7)$$

with $b = sa\overline{s}$ and the group operation $(s_1,a_1)\circ(s_2,a_2) = (s_1s_2,(s_1a_2\overline{s_1})\times a_1)$, where $s_1,s_2 \in \mathrm{Spin}(n)$ and $a_1,a_2 \in B^n$.

By the results obtained in section 5 and lemma 4.3 we can restrict the point $a \in B^n$ to the bidimensional parameter $d = (r\sin\phi, 0,\cdots,0, r\cos\phi)$, with $r \in [0,1[$ and $\phi \in [0,\pi]$. In fact we can describe an arbitrary point $a \in B^n$ in terms of a point of the form d and a convenient rotation, as e.g.

$$\begin{aligned} a &= s_1\cdots s_{n-2}s_{n-1}\, r\, \mathrm{e}_n\, \overline{s_{n-1}}\,\overline{s_{n-2}}\cdots\overline{s_1} \\ &= \hat{s}d\overline{\hat{s}}, \qquad (6.8)\end{aligned}$$

with $\hat{s} = s_1\cdots s_{n-2} \in \mathrm{Spin}(n)$ and $d = s_{n-1}\, r\, \mathrm{e}_n\, \overline{s_{n-1}}$ (cf. lemma (4.3)). We remark that, as seen before, the parameters $\theta_1,\ldots,\theta_{n-2}$ only give us information about the localization of the attractor point and this information can also be obtained by the action of the Spin group.

In this way, $D_1(d)$ is our dilatation operator, which will be used from now on.

With respect to the square integrability of our representation we have the following theorem (see [15])

Theorem 6.1. *A function $\psi \in L_2(S^{n-1},\mathbb{R}_{0,n})$ is admissible if there exists a finite constant $c > 0$ such that*

$$\sum_m \int_0^\pi \int_0^1 |a_k^{(m)}(d)|_0^2 \frac{r}{(1-r^2)^n} dr d\phi < c \qquad (6.9)$$

where $a_k^{(m)}(d) =< H_k^{(m)}, D_1(d)\psi >$ denotes the Fourier coefficients associated to the orthonormal basis of spherical harmonics $\{H_k^{(i)}, i = 1,\cdots,N(n,k)\}_{k=0}^\infty$.

As a necessary (and almost sufficient) condition for the admissibility we have (see [15]):

Proposition 6.2. *Let $\psi \in L_2(S^{n-1})$ be a function with support on a given spherical calotte $\mathcal{U}_h$. If ψ is an admissible function then it necessarily satisfies the condition*

$$\int_{\mathcal{U}_h} \frac{\psi(y)}{(1+\sin\phi\, y_1 + \cos\phi\, y_n)^{\frac{n-1}{2}}} dS_y = 0, \qquad (6.10)$$

$\forall\, \phi \in [\arccos(h),\pi]$.

It is difficult to prove the existence of wavelets if we consider the bidimensional parameter d. However we can fix an angle $\phi \in [0, \pi/2]$ and consider the one dimensional subgroup of Möbius transformations obtained by the parameter $d_\phi = (t \sin\phi, 0, \ldots, 0, t\cos\phi)$, with $-1 < t < 1$. Then we can reformulate our admissibility condition (6.9).

Theorem 6.3. *A function $\psi \in L_2(S^{n-1}, \mathbb{R}_{0,n})$ is admissible if there exists a finite constant $c > 0$ such that*

$$\sum_m \int_{-1}^{1} |a_k^{(m)}(d_\phi)|_0^2 \frac{dt}{(1-t^2)^n} < c \tag{6.11}$$

where $a_k^{(m)}(d_\phi) =< H_k^{(m)}, D_1(d_\phi)\psi >$ denotes the Fourier coefficients.

Also the necessary condition (6.10) can now be stated as

$$\int_{\mathcal{U}_h} \frac{\psi(y)}{(|1-\omega y|^2)^{\frac{n-1}{2}}} dS_y = 0 \tag{6.12}$$

where $\omega = (\sin\phi, 0, \ldots, 0, \cos\phi)$ and $\mathcal{U}_h$ is chosen such that $-\cos\phi < h < \cos\phi$. With this condition on the calotte the operator $D_1(d_\phi)$ is a conformal dilatation/contraction operator on the sphere (see [15]). This is a zero mean condition and in the case $\phi = 0$ we obtain the same necessary condition considered in [2], $\int_{S^{n-1}} \frac{\psi(y)}{(1+y_n)^{\frac{n-1}{2}}} dS_y = 0$, which is the exact equivalent of the usual necessary condition for Euclidean wavelets, $\int \psi(x) d^{n-1}x = 0$ (see [2]). One simple way to construct wavelets to our case is to consider a function ψ with compact support on $\mathcal{U}_h$ such that $\int_{U_h} \psi(y) dS_y = 0$. Then the function $\psi_1(y) = (|1-\omega y|^2)^{\frac{n-1}{2}} \psi(y)$, $y \in \mathcal{U}_h$ is an (almost) admissible function i.e., a wavelet.

For an admissible function ψ and $f \in L_2(S^{n-1})$ we define the continuous wavelet transform on the sphere as

$$V_\psi f(s, d_\phi) :=< \tilde{\pi}_1(s, d_\phi)\psi, f >_{L_2} .$$

Thus V_ψ can be inverted on its range by its adjoint V_ψ^* given by

$$V_\psi^* F(x) =$$

$$\int_{Spin(n)} \int_{-1}^{1} \big(\tilde{\pi}_1(s, d_\phi)\psi\big)(x) F(s, d_\phi)\, d\mu(s) \frac{dt}{(1-t^2)^n}$$

In the case of the Hardy space H^2 we consider two different operators that preserve the monogenicity of the function, namely,

$$R_2(s) f(x) = s f(\bar{s} x s), \ \ s \in \mathrm{Spin}(n) \tag{6.13}$$

and

$$D_2(a) f(x) =$$

$$(1-|a|^2)^{\frac{n-1}{2}} \frac{1 - xa}{|1-ax|^n} f(\varphi_{-a}(x)). \tag{6.14}$$

Therefore we have the representation

$$\pi_2(s,a)f(x) = R_2(s) \circ D_2(a)f(x) =$$

$$= (1-|a|^2)^{\frac{n-1}{2}} s \frac{1-\overline{s}xsa}{|1-a\overline{s}xs|^n} f(\varphi_{-a}(\overline{s}xs)) \tag{6.15}$$

This representation is equivalent to

$$\tilde{\pi}_2(s,b)f(x) = (1-|a|^2)^{\frac{n-1}{2}} \frac{1-xb}{|1-bx|^n} sf(\overline{s}\varphi_{-b}(x)s) \tag{6.16}$$

with $b = sa\overline{s}$.

Once again we restrict the parameter a to a parameter d_ϕ in order to have a conformal dilatation/contraction operator on the sphere.

By considering an orthonormal basis of inner spherical monogenics $(P(k)f)_{k\in\mathbb{N}}$ for the Hardy space H^2 we can derive an admissibility condition for wavelets in this space.

Theorem 6.4. *A function $\psi \in H^2$ is admissible if there exists a finite constant $c > 0$ such that*

$$\sum_m \int_{-1}^{1} |a_k^{(m)}(d_\phi)|_0^2 \frac{dt}{(1-t^2)^n} < c \tag{6.17}$$

where $a_k^{(m)}(d_\phi) =< P_k^{(m)}, D_2(d_\phi)\psi >_{L_2}$ are the Fourier coefficients of $D_2(d_\phi)\psi$.

In the case of the Hardy space our necessary condition reads as follows.

Proposition 6.5. *Let $\psi \in H^2$ be a function with compact support, i.e. ψ lives on a spherical calotte $\mathcal{U}_h$ such that $-\cos(\phi) < h < \cos(\phi)$. If ψ is an admissible function then necessarily it satisfies the condition*

$$\int_{\mathcal{U}_h} \frac{1-\omega y}{|1-\omega y|^n} \psi(y) dS_y = 0, \tag{6.18}$$

where $\omega = (\sin\phi, 0, \ldots, 0, \cos\phi)$.

Finally we define for an admissible $\psi \in H^2$ and $f \in H^2$ the continuous wavelet transform as

$$W_\psi f(s, d_\phi) :=< \tilde{\pi}_2(s, d_\phi)\psi, f >_{H_2} .$$

Let us remark that in this case $W_\psi f$ is monogenic in the unit ball.

We would like to remark that although $\psi \equiv 1$ is not an admissible function if we use it in this CWT we recover the Cauchy integral formula in Clifford Analysis up to a constant factor.

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Clifford-Jacobi Polynomials and the Associated Continuous Wavelet Transform in Euclidean Space

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Abstract. Specific wavelet kernel functions for a continuous wavelet transform in Euclidean space are presented within the framework of Clifford analysis. These multi-dimensional wavelets are constructed by taking the Clifford-monogenic extension to $\mathbb{R}^{m+1}$ of specific functions in $\mathbb{R}^m$ generalizing the traditional Jacobi weights. The notion of Clifford-monogenic function is a direct higher dimensional generalization of that of holomorphic function in the complex plane. Moreover, crucial to this construction is the orthogonal decomposition of the space of square integrable functions into the Hardy space $H^2(\mathbb{R}^m)$ and its orthogonal complement. In this way a nice relationship is established between the theory of the Clifford Continuous Wavelet Transform on the one hand, and the theory of Hardy spaces on the other hand. Furthermore, also new multi-dimensional polynomials, the so-called Clifford-Jacobi polynomials, are obtained.

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1. Introduction

The wavelet transform has become quite a standard tool in numerous research and application domains and its popularity has increased rapidly over the last decades (see e.g. [1, 2, 3]).

Wavelets on the real line constitute a family of functions $\psi_{a,b}$ derived from one single function ψ, called the mother wavelet, by change of scale a (i.e. by dilation) and by change of position b (i.e. by translation):

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right) \quad , \quad a > 0 \quad , \quad b \in \mathbb{R} \quad .$$

In wavelet theory some conditions on the mother wavelet ψ have to be imposed. We request ψ to be an L_2-function (finite energy signal) which is well localized both in the time domain and in the frequency domain. Moreover ψ has to satisfy the so-called admissibility condition:

$$C_\psi := \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(u)|^2}{|u|} \, du < +\infty \quad ,$$

where $\widehat{\psi}$ denotes the Fourier transform of ψ. In the case where ψ is also in L_1, this admissibility condition implies

$$\int_{-\infty}^{+\infty} \psi(x) \ dx = 0 \quad .$$

In other words: ψ must be an oscillating function, which explains its qualification as "wavelet".
In practice, applications impose additional requirements, among which a given number of vanishing moments:

$$\int_{-\infty}^{+\infty} x^n \ \psi(x) \ dx = 0 \quad , \quad n = 0, 1, \ldots, N \quad .$$

This means that the corresponding Continuous Wavelet Transform (abbreviated CWT):

$$F(a,b) = < \psi_{a,b}, f > = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \overline{\psi} \left(\frac{x-b}{a} \right) \ f(x) \ dx$$

will filter out polynomial behaviour of the signal f up to degree N, making it adequate at detecting singularities.
When considering two L_2-functions f and g with respective CWT-images F and G, the following inner product in the space of transforms may be introduced:

$$[F,G] = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} \overline{F}(a,b) \ G(a,b) \ \frac{da}{a^2} \ db \quad .$$

Taking into account the above mentioned admissibility condition for the mother wavelet ψ, the corresponding Parseval formula is readily obtained:

$$[F,G] = < f, g > \quad .$$

In other words, as a consequence of the admissibility condition the CWT is an isometry from the space of signals into the space of transforms.
The CWT may be extended to higher dimensions while still enjoying the same properties as in the one dimensional case. These higher dimensional CWTs typically originate as tensor products of one dimensional phenomena. However also the non-separable treatment of two dimensional wavelets should be mentioned (see [4]).

The classical wavelet theory can be extended to the Clifford analysis setting. Clifford analysis may be regarded as a direct and elegant generalization to higher dimension of the theory of the holomorphic functions in the complex plane, centred

around the notion of monogenic function, i.e. a null solution of the Dirac operator (Section 2).
The first step in the construction of multi-dimensional Clifford-wavelets, is the introduction of new polynomials, generalizing classical orthogonal polynomials on the real line to the Clifford analysis setting. Their construction rests upon a specific Clifford analysis technique, the so-called Cauchy-Kowalewskaia extension of a real-analytic function in $\mathbb{R}^m$ to a monogenic function in $\mathbb{R}^{m+1}$. One starts from a real-analytic function in an open connected domain in $\mathbb{R}^m$, as an analogue of the classical weight function. The new Clifford algebra-valued polynomials are then generated by the Cauchy-Kowalewskaia extension of this weight function. For these polynomials a recurrence relation and a Rodrigues formula are established. This Rodrigues formula together with Stokes's theorem lead to an orthogonality relation of the new Clifford-polynomials. From this orthogonality relation we select candidates for mother wavelets and show that these candidates indeed may serve as kernel functions for a multi-dimensional continuous wavelet transform if they satisfy certain conditions (see [5]).
We have applied the above technique and have constructed in this way Clifford-wavelets on the basis of Clifford generalizations of the Hermite polynomials, the Laguerre polynomials and the Gegenbauer polynomials (see [6, 7, 8, 9, 10]).

In this paper the remaining class of Clifford algebra-valued basic wavelet functions is presented, namely the one based on a Clifford generalization of the classical Jacobi polynomials on the real line. As a Clifford generalization of the Jacobi weight function, we take the Clifford algebra-valued function $F(\underline{x}) = (1+\underline{x})^{\alpha}\,(1-\underline{x})^{\beta}$ with $\alpha, \beta \in \mathbb{R}$. The Cauchy-Kowalewskaia extension of this weight function generates the so-called general Clifford-Jacobi polynomials (Section 3). For these polynomials a recurrence relation and a Rodrigues formula are established, but no orthogonality relation can be derived. However the special case $\alpha = \beta + 1$ does yield orthogonal polynomials, which are called the special Clifford-Jacobi polynomials (subsection 4.1). These polynomials are the appropriate building blocks for the so-called Clifford-Jacobi wavelets (subsection 4.2). However, these wavelets do not satisfy the mother wavelet conditions established in [5]. Nevertheless we are able to use them as kernel functions for a multi-dimensional Clifford CWT. To that end we are forced to use the orthogonal decomposition of the space of square integrable functions into the Hardy space $H^2(\mathbb{R}^m)$ and its orthogonal complement (subsection 4.3). In this way a nice relationship is established between the theory of the Clifford CWT on the one hand, and the theory of Hardy spaces on the other hand.

2. The Clifford Toolbox

Clifford analysis (see e.g. [11] and [12]) offers a function theory which is a higher dimensional analogue of the theory of the holomorphic functions of one complex variable. Consider functions defined in $\mathbb{R}^m$ $(m > 1)$ and taking values in the

Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for $\mathbb{R}_m$ or $\mathbb{C}_m$ is given by $(e_A \; : \; A \subset \{1, \ldots, m\})$ where $e_\emptyset = 1$ is the identity element.
The non-commutative multiplication in the Clifford algebra is governed by the rules:

$$e_j e_k + e_k e_j = -2 \; \delta_{j,k} \quad , \quad j, k = 1, \ldots, m \quad .$$

Conjugation is defined as the anti-involution for which

$$\overline{e_j} = -e_j \quad , \quad j = 1, \ldots, m \quad .$$

In the case of $\mathbb{C}_m$, the Hermitian conjugate of an element $\lambda = \sum_A \lambda_A \, e_A \; (\lambda_A \in \mathbb{C})$ is defined by

$$\lambda^\dagger = \sum_A \lambda_A^c \; \overline{e_A}$$

where λ_A^c denotes the complex conjugate of λ_A.

In what follows $\mathbb{R}_m^k$ denotes the subspace of k-vectors, i.e. the space spanned by the products of k different basis vectors.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^m e_j x_j \quad ,$$

whereas the Euclidean space $\mathbb{R}^{m+1}$ is identified with $\mathbb{R}_m^0 \oplus \mathbb{R}_m^1$ by identifying $(x_0, x_1, \ldots, x_m)$ with the paravector $x_0 + \underline{x}$.

The Spin-group

$$Spin(m) = \left\{ s = \underline{\omega}_1 \ldots \underline{\omega}_{2\ell} \; ; \; \underline{\omega}_j \in S^{m-1}, \; j = 1, \ldots, 2\ell, \; \ell \in \mathbb{N} \right\} \quad ,$$

where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$, is a two-fold covering group of the rotation group $SO(m)$. For each $T \in SO(m)$ there exists $s \in Spin(m)$ such that $T(\underline{x}) = s\underline{x}\overline{s} = (-s) \; \underline{x} \; (-\overline{s})$, for all $\underline{x} \in \mathbb{R}^m$.

An $\mathbb{R}_m$- or $\mathbb{C}_m$-valued function $F(x_1, \ldots, x_m)$, respectively $G(x_0, x_1, \ldots, x_m)$, is called left monogenic in an open region of $\mathbb{R}^m$, respectively $\mathbb{R}^{m+1}$, if in that region:

$$\partial_{\underline{x}} F = 0 \quad , \quad \text{respectively} \quad (\partial_{x_0} + \partial_{\underline{x}}) G = 0 \quad .$$

Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$:

$$\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j} \quad ,$$

whereas $\partial_{x_0} + \partial_{\underline{x}}$ is the Cauchy-Riemann operator in $\mathbb{R}^{m+1}$.
The notion of right monogenicity is defined in a similar way by letting act the Dirac operator, or the Cauchy-Riemann operator, from the right.

A left, respectively right, monogenic homogeneous polynomial P_k of degree k ($k \geq 0$) in $\mathbb{R}^m$ is called a left, respectively right, inner spherical monogenic of order k.

If $\underline{\Omega} \subset \mathbb{R}^m$ is open, then an open neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$ is said to be x_0-normal, if for each $x \in \Omega$ the line segment $\{x + t\ ;\ t \in \mathbb{R}\} \cap \Omega$ is connected and contains exactly one point in $\underline{\Omega}$.
Considering $\mathbb{R}^m$ as the hyperplane $x_0 = 0$ in $\mathbb{R}^{m+1}$, a real-analytic function $f(\underline{x})$ in an open connected domain $\underline{\Omega}$ in $\mathbb{R}^m$ can be uniquely extended to a monogenic function $f^*(x_0, \underline{x})$ in an open connected and x_0-normal neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$. This so-called Cauchy-Kowalewskaia (CK-) extension of $f(\underline{x})$ is given by

$$f^*(x_0, \underline{x}) = \sum_{\ell=0}^{\infty} (-1)^\ell\ \frac{x_0^\ell}{\ell!}\ \partial_{\underline{x}}^\ell f(\underline{x}) \quad . \tag{2.1}$$

Introducing the fundamental solution E of the Cauchy-Riemann operator, given by

$$E(x_0, \underline{x}) = \frac{1}{A_{m+1}} \frac{x_0 - \underline{x}}{|x_0 + \underline{x}|^{m+1}}$$

where A_{m+1} stands for the area of the unit sphere S^m in $\mathbb{R}^{m+1}$, we may define for a square integrable function $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$, its Cauchy integral in the half spaces $\mathbb{R}^{m+1}_{\pm} = \{(x_0, \underline{x}) \in \mathbb{R}^{m+1}\ :\ x_0 \underset{<}{>} 0\}$ by

$$\mathcal{C}[f](x_0, \underline{x}) = E(x_0,\ .\ ) * f(\ .\ )(\underline{x}) = \int_{\mathbb{R}^m} E(x_0, \underline{x} - \underline{y})\ f(\underline{y})\ dV(\underline{y}) \quad ,$$

for $x_0 \neq 0$, where $dV(\underline{y})$ stands for the Lebesgue measure on $\mathbb{R}^m$. This Cauchy integral is a linear isomorphism between $L_2(\mathbb{R}^m, dV(\underline{x}))$ and the Hardy space $H^2(\mathbb{R}^{m+1}_+)$ (see for e.g. [13]).
Two projection operators may be defined by considering the $L_2(\mathbb{R}^m, dV(\underline{x}))$ non-tangential boundary values for $x_0 \to 0+$ and $x_0 \to 0-$ of the Cauchy integral:

$$\mathbb{P}^+[f] = \lim_{\substack{x_0 \to 0 \\ >}} \mathcal{C}[f](x_0, \underline{x}) = \frac{1}{2} f(\underline{x}) + \frac{1}{2} H[f](\underline{x})$$

and

$$\mathbb{P}^-[f] = -\lim_{\substack{x_0 \to 0 \\ <}} \mathcal{C}[f](x_0, \underline{x}) = \frac{1}{2} f(\underline{x}) - \frac{1}{2} H[f](\underline{x}) \quad ,$$

where $H[f]$ denotes the Hilbert transform of the function f. This yields the orthogonal decomposition of the space of square integrable functions into the Hardy space $H^2(\mathbb{R}^m)$ and its orthogonal complement:

$$L_2(\mathbb{R}^m, dV(\underline{x})) = H^2(\mathbb{R}^m) \oplus H^2(\mathbb{R}^m)^\perp \quad .$$

In the sequel, the Fourier transform of f will be denoted by $\mathcal{F}[f]$; it is defined by

$$\mathcal{F}[f](\underline{\xi}) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\ f(\underline{x})\ dV(\underline{x}).$$

The Fourier transform of the Hilbert transform is given by

$$\mathcal{F}\big[H[f]\big](\underline{\xi}) = i\underline{\eta}\ \mathcal{F}[f](\underline{\xi})$$

where we have used spherical co-ordinates in frequency space given by

$$\underline{\xi} = \rho\ \underline{\eta} \quad , \quad \rho = |\underline{\xi}| \in [0, +\infty[\quad , \quad \underline{\eta} \in S^{m-1} \quad .$$

Hence, the orthogonal decomposition of an $L_2(\mathbb{R}^m, dV(\underline{x}))$ - function f :

$$f = \mathbb{P}^+[f] + \mathbb{P}^-[f]$$

reads in frequency space

$$\mathcal{F}[f] = \frac{1}{2}(1 + i\underline{\eta})\ \mathcal{F}[f] + \frac{1}{2}(1 - i\underline{\eta})\ \mathcal{F}[f] \quad .$$

Here the so-called Clifford-Heaviside functions

$$P^+ = \frac{1}{2}\left(1 + i\underline{\eta}\right) \quad , \quad P^- = \frac{1}{2}\left(1 - i\underline{\eta}\right) \quad , \quad \underline{\eta} \in S^{m-1} \quad ,$$

appear; they were introduced independently by Sommen in [14] and McIntosh in [15] and [16]. They are self-adjoint mutually orthogonal idempotents:

$$P^+ + P^- = 1 \ ; \ P^+P^- = P^-P^+ = 0 \ ; \ (P^+)^2 = P^+ \ ; \ (P^-)^2 = P^- \quad . \tag{2.2}$$

Furthermore, they satisfy

$$i\underline{\xi}\ P^\pm = \pm|\underline{\xi}|\ P^\pm \quad \text{or} \quad \underline{\xi}\ P^\pm = \mp i\ |\underline{\xi}|\ P^\pm \quad . \tag{2.3}$$

3. The General Clifford-Jacobi Polynomials

As a generalization to Clifford analysis of the classical Jacobi weight function, we take the Clifford algebra-valued weight function $F(\underline{x}) = (1 + \underline{x})^\alpha(1 - \underline{x})^\beta$ with $\alpha, \beta \in \mathbb{R}$. Keeping in mind the properties (2.2) and (2.3) of the Clifford-Heaviside functions, we define the factor $(1 + \underline{x})^\alpha$ as follows:

$$(1 + \underline{x})^\alpha := (1 - ir)^\alpha\ P^+ + (1 + ir)^\alpha\ P^-$$

with $r = |\underline{x}|$. Note that both terms $(1 \pm ir)^\alpha$ are well-defined for each $r \in [0, +\infty[$. Indeed, the complex-valued function $(1 + z)^\alpha$ $(\alpha \in \mathbb{R})$ is defined in the whole complex plane except for a branch cut, which can be chosen along the negative real axis from -1 to $-\infty$.
Similarly, we define

$$(1 - \underline{x})^\beta := (1 + ir)^\beta\ P^+ + (1 - ir)^\beta\ P^- \quad .$$

This method for defining the factors $(1 + \underline{x})^\alpha$ and $(1 - \underline{x})^\beta$ was also used in [15] and [17].
Hence, the weight function may also be written as

$$F(\underline{x}) = (1 - ir)^\alpha\ (1 + ir)^\beta\ P^+ + (1 + ir)^\alpha\ (1 - ir)^\beta\ P^- \quad .$$

Note that the second term is the complex conjugate of the first one, which is in accordance with the fact that $F(\underline{x})$ takes its values in the real Clifford algebra $\mathbb{R}_m$.

The general Clifford-Jacobi polynomials, denoted by $J^{+}_{\ell,\alpha,\beta}(\underline{x})$ and $J^{-}_{\ell,\alpha,\beta}(\underline{x})$, are generated by the CK-extension $F^*(x_0,\underline{x})$ of the weight function $F(\underline{x})$:

$$\begin{aligned}
& F^*(x_0,\underline{x}) \\
= \; & \sum_{\ell=0}^{\infty}(-1)^{\ell}\ \frac{x_0^{\ell}}{\ell!}\ \partial_{\underline{x}}^{\ell}F(\underline{x}) \\
= \; & \sum_{\ell=0}^{\infty}\frac{x_0^{\ell}}{\ell!}(1-ir)^{\alpha-\ell}(1+ir)^{\beta-\ell}r^{-\ell}\left[J^{+}_{\ell,\alpha,\beta}(\underline{x})\ P^{+}+J^{-}_{\ell,\alpha,\beta}(\underline{x})\ P^{-}\right] \\
& + \sum_{\ell=0}^{\infty}\frac{x_0^{\ell}}{\ell!}(1+ir)^{\alpha-\ell}(1-ir)^{\beta-\ell}r^{-\ell}\left[(J^{-}_{\ell,\alpha,\beta}(\underline{x}))^{c}\ P^{+}+(J^{+}_{\ell,\alpha,\beta}(\underline{x}))^{c}\ P^{-}\right] \ .
\end{aligned}$$

By definition, we have

$$F^*(0,\underline{x}) = (1+\underline{x})^{\alpha}\ (1-\underline{x})^{\beta} \quad ,$$

which yields

$$J^{+}_{0,\alpha,\beta}(\underline{x}) = 1 \quad \text{and} \quad J^{-}_{0,\alpha,\beta}(\underline{x}) = 0 \quad .$$

Furthermore, the monogenicity of $F^*(x_0,\underline{x})$ leads to the recurrence relation:

$$\begin{aligned}
& J^{+}_{\ell+1,\alpha,\beta}(\underline{x})\ P^{+}+J^{-}_{\ell+1,\alpha,\beta}(\underline{x})\ P^{-} \\
= \; & i\big((\alpha-\ell)(1+ir)-(\beta-\ell)(1-ir)\big)\ \underline{x}\ \big(J^{+}_{\ell,\alpha,\beta}(\underline{x})\ P^{+}+J^{-}_{\ell,\alpha,\beta}(\underline{x})\ P^{-}\big) \\
& + i\ell(1+|\underline{x}|^2)\ \big(-J^{+}_{\ell,\alpha,\beta}(\underline{x})\ P^{+}+\ J^{-}_{\ell,\alpha,\beta}(\underline{x})\ P^{-}\big) \\
& - (1+|\underline{x}|^2)\ r\ \partial_{\underline{x}}\big(J^{+}_{\ell,\alpha,\beta}(\underline{x})\ P^{+}+J^{-}_{\ell,\alpha,\beta}(\underline{x})\ P^{-}\big) \quad ,
\end{aligned}$$

from which the general Clifford-Jacobi polynomials can be computed recursively. It appears that $J^{\pm}_{\ell,\alpha,\beta}(\underline{x})$ is a polynomial of degree 2ℓ in the variable $\underline{x}$.
One also has the Rodrigues formula:

$$\begin{aligned}
& \partial_{\underline{x}}^{\ell}\big((1+\underline{x})^{\alpha}(1-\underline{x})^{\beta}\big) \\
= \; & (-1)^{\ell}r^{-\ell}\Bigg((1-ir)^{\alpha-\ell}(1+ir)^{\beta-\ell}\ \big(J^{+}_{\ell,\alpha,\beta}(\underline{x})\ P^{+}+J^{-}_{\ell,\alpha,\beta}(\underline{x})\ P^{-}\big) \\
& + (1+ir)^{\alpha-\ell}\ (1-ir)^{\beta-\ell}\ \big(J^{+}_{\ell,\alpha,\beta}(\underline{x})\ P^{+}+J^{-}_{\ell,\alpha,\beta}(\underline{x})\ P^{-}\big)^{c}\Bigg) \quad .
\end{aligned}$$

Remark 3.1. Combining the above Rodrigues formula with Stokes's theorem does not yield an orthogonality relation for the general Clifford-Jacobi polynomials w.r.t. the weight function $F(\underline{x})$. However, it is possible to construct orthogonal polynomials in the special case where $\alpha = \beta + 1$ (see next section).

4. The Special Clifford-Jacobi Polynomials and Associated CWT

4.1. The Special Clifford-Jacobi Polynomials

In this section we consider the special case where $\alpha = \beta + 1$ ($\beta \in \mathbb{R}$), i.e. we consider the Clifford algebra-valued weight function:

$$\begin{aligned} F(\underline{x}) &= (1+\underline{x})^{\beta+1}\ (1-\underline{x})^{\beta} \\ &= (1+\underline{x})\ (1+|\underline{x}|^2)^{\beta} \quad . \end{aligned}$$

Natural powers of the Dirac operator acting on the weight function can be written as

$$\partial_{\underline{x}}^{\ell}\Big((1+|\underline{x}|^2)^{\beta}\ (1+\underline{x})\Big) = (1+|\underline{x}|^2)^{\beta-\ell}\ (1+\underline{x})\ J_{\ell,\beta}(\underline{x})$$

with $J_{\ell,\beta}(\underline{x})$ a polynomial of degree ℓ in the vector variable $\underline{x}$. The CK-extension of $F(\underline{x})$ then takes the form

$$F^*(x_0,\underline{x}) = \sum_{\ell=0}^{\infty} \frac{x_0^{\ell}}{\ell!}\ (1+|\underline{x}|^2)^{\beta-\ell}\ (1+\underline{x})\ J_{\ell,\beta}(\underline{x}) \quad .$$

Remark 4.1. Note that if we take more generally $\alpha = \beta + n$ ($n \in \mathbb{N} \setminus \{1\}$), i.e. we consider the weight function

$$G(\underline{x}) = (1+\underline{x})^{n}\ (1+|\underline{x}|^2)^{\beta} \quad , \quad n = 2,3,4,\ldots$$

then we can also write

$$\partial_{\underline{x}}^{\ell}\big((1+|\underline{x}|^2)^{\beta}\ (1+\underline{x})^{n}\big) = (1+|\underline{x}|^2)^{\beta-\ell}\ (1+\underline{x})\ J_{\ell,\beta,n}(\underline{x})$$

with $J_{\ell,\beta,n}(\underline{x})$ a polynomial of degree $n+\ell-1$ in $\underline{x}$.
However in order to obtain, by means of the CK-extension technique, orthogonal polynomials w.r.t. the weight function $G(\underline{x})$ we should have had a relation of the form

$$\partial_{\underline{x}}^{\ell}\big((1+|\underline{x}|^2)^{\beta}\ (1+\underline{x})^{n}\big) = (1+|\underline{x}|^2)^{\beta-\ell}\ (1+\underline{x})^{n}\ J^*_{\ell,\beta,n}(\underline{x})$$

with $J^*_{\ell,\beta,n}(\underline{x})$ a polynomial of degree ℓ in $\underline{x}$.
Hence, the case $\alpha = \beta+1$ is the only case where the CK-extension technique leads to orthogonal polynomials w.r.t. the weight function.
Note that the same situation occurred when studying the generalized Clifford-Jacobi polynomials on the open unit ball $B(1)$ of Euclidean space (see [18]). These Clifford algebra-valued polynomials, which were constructed by means of a different approach, are orthogonal in $B(1)$ w.r.t. the weight function $(1+i\underline{x})^{\alpha}\ (1-i\underline{x})^{\beta}$ ($\alpha, \beta > -1$). There the case $\beta = \alpha+1$ is also special. Indeed, only when $\beta = \alpha+1$, it is possible to obtain an explicit recurrence relation for the generalized Clifford-Jacobi polynomials.

From the monogenicity relation of $F^*(x_0,\underline{x})$, we derive the recurrence relation:

$$J_{\ell+1,\beta}(\underline{x}) = 2(\ell-\beta)\underline{x}\ J_{\ell,\beta}(\underline{x}) + \ (\underline{x}-1)\ \partial_{\underline{x}}\big((1+\underline{x})\ J_{\ell,\beta}(\underline{x})\big) \quad .$$

As $J_{0,\beta}(\underline{x}) = 1$, we thus obtain

$$\begin{array}{rcl} J_{1,\beta}(\underline{x}) & = & m - (2\beta + m)\underline{x} \\ J_{2,\beta}(\underline{x}) & = & -2\beta m - 4\beta\underline{x} + 2\beta(2\beta + m)\underline{x}^2 \\ J_{3,\beta}(\underline{x}) & = & -2\beta m(m+2) + 2\beta(2\beta m + 4\beta - 4 + m^2)\underline{x} \\ & & +2\beta(-4 + 4\beta + 2\beta m + m^2)\underline{x}^2 + 2\beta(2\beta + m)(2 - 2\beta - m)\underline{x}^3 \\ & & \text{and so on.} \end{array}$$

From the explicit formula (2.1) for the CK-extension, we obtain the Rodrigues formula:

$$J_{\ell,\beta}(\underline{x}) = (-1)^\ell \ (1 + |\underline{x}|^2)^{\ell-\beta} \ (1 + \underline{x})^{-1} \ \partial_{\underline{x}}^\ell \Big((1 + \underline{x}) \ (1 + |\underline{x}|^2)^\beta \Big)$$

which together with Stokes's theorem yields the following orthogonality relation.

Theorem 4.2. *Whenever $\ell < t < (-2\beta - m)/2$, one has the orthogonality relation*

$$\int_{\mathbb{R}^m} J^\dagger_{\ell,\beta+\ell}(\underline{x}) \ J_{t,\beta+t}(\underline{x}) \ (1 + \underline{x}) \ (1 + |\underline{x}|^2)^\beta \ dV(\underline{x}) = 0 \quad .$$

4.2. The Clifford-Jacobi Wavelets

Note that Theorem 4.2 implies that for $0 < t < (-2\beta - m)/2$

$$\int_{\mathbb{R}^m} J_{t,\beta+t}(\underline{x}) \ (1 + \underline{x}) \ (1 + |\underline{x}|^2)^\beta \ dV(\underline{x}) = 0 \quad .$$

Consequently, the $L_1 \cap L_2$-functions

$$\begin{array}{rcl} \psi_{t,\beta}(\underline{x}) & = & J_{t,\beta+t}(\underline{x}) \ (1 + \underline{x}) \ (1 + |\underline{x}|^2)^\beta \\ & = & (-1)^t \ \partial_{\underline{x}}^t \Big((1 + \underline{x}) \ (1 + |\underline{x}|^2)^{\beta+t} \Big) \end{array}$$

have zero momentum. In Section 4.3 we will show that they can be used as mother wavelets; we call them the Clifford-Jacobi wavelets. Note that the condition $0 < t < (-2\beta - m)/2$ forces us to make the restriction $\beta < -m/2$.

These Clifford-Jacobi wavelets are invariant under the rotation group $Spin(m)$, which means that

$$s \ \psi_{t,\beta}(\overline{s}\underline{x}s) \ \overline{s} = \psi_{t,\beta}(\underline{x}) \quad , \quad s \in Spin(m) \quad .$$

Furthermore, the wavelets $\psi_{t,\beta}(\underline{x})$ have vanishing moments if the condition $2\beta < -m - t - 1$ is fulfilled, as is shown in the next proposition.

Proposition 4.3. *If $2\beta < -m - t - 1$, the Clifford-Jacobi wavelet $\psi_{t,\beta}(\underline{x})$ has vanishing moments:*

$$\int_{\mathbb{R}^m} \underline{x}^j \ \psi_{t,\beta}(\underline{x}) \ dV(\underline{x}) = 0$$

for $0 \leq j < -m - t - 2\beta - 1$ and $j < t$.

Moreover, the Fourier transform of the Clifford-Jacobi wavelets takes the form:

$$\begin{aligned}\mathcal{F}[\psi_{t,\beta}](\underline{\xi}) &= (-i)^t \frac{2^{\beta+t+1}}{\Gamma(-\beta-t)} \, |\underline{\xi}|^{-m/2-\beta-t-1} \underline{\xi}^t \\ &\quad \left\{ |\underline{\xi}| \, K_{m/2+\beta+t}(|\underline{\xi}|) - i \, K_{m/2+\beta+t+1}(|\underline{\xi}|) \, \underline{\xi} \right\} \end{aligned} \tag{4.1}$$

with $K_\nu(t)$ the modified Bessel function of the second kind, also called Macdonald function.

4.3. The Clifford-Jacobi CWT

The Clifford-Jacobi wavelets do not satisfy the mother wavelet conditions of the general Clifford CWT theory established in [5]. Indeed, in view of (4.1) we see that

$$\begin{aligned}&\mathcal{F}[\psi_{t,\beta}](\underline{\xi}) \, \left(\mathcal{F}[\psi_{t,\beta}](\underline{\xi})\right)^\dagger \\ &= \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 |\underline{\xi}|^{-m-2\beta} \left(\left(K_{m/2+\beta+t}(|\underline{\xi}|)\right)^2 + \left(K_{m/2+\beta+t+1}(|\underline{\xi}|)\right)^2 \right) \\ &\quad - 2i \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 |\underline{\xi}|^{-m-2\beta} K_{m/2+\beta+t+1}(|\underline{\xi}|) \, K_{m/2+\beta+t}(|\underline{\xi}|) \, \frac{\underline{\xi}}{|\underline{\xi}|} \end{aligned} \tag{4.2}$$

which is in fact not radial symmetric (i.e. only depending on $|\underline{\xi}|$) as it should be.

Nevertheless we are able to use the Clifford-Jacobi wavelets as kernel functions for a multi-dimensional Clifford CWT. For this purpose, we are forced to decompose each square integrable function $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ into its Hardy components (see Section 2)

$$f = \mathbb{P}^+[f] + \mathbb{P}^-[f]$$

with

$$\mathbb{P}^+[f] = \frac{1}{2}(f + H[f]) \in H^2(\mathbb{R}^m)$$

and

$$\mathbb{P}^-[f] = \frac{1}{2}(f - H[f]) \in H^2(\mathbb{R}^m)^\perp \quad .$$

In what follows, we briefly denote $f^\pm = \mathbb{P}^\pm[f]$.
In this way we define, still for $0 < t < (-2\beta - m)/2$, the 'half' Clifford-Jacobi CWTs by

$$\begin{aligned} T^\pm_{t,\beta}[f^\pm](a,\underline{b}) &= F^\pm_{t,\beta}(a,\underline{b}) \\ &= < \psi^{a,\underline{b}}_{t,\beta} \, , \, f^\pm > \\ &= \int_{\mathbb{R}^m} \left(\psi^{a,\underline{b}}_{t,\beta}(\underline{x})\right)^\dagger \, f^\pm(\underline{x}) \, dV(\underline{x}) \; , \end{aligned}$$

where the continuous family of wavelets

$$\psi^{a,\underline{b}}_{t,\beta}(\underline{x}) = \frac{1}{a^{m/2}} \, \psi_{t,\beta}\left(\frac{\underline{x}-\underline{b}}{a}\right) \quad ,$$

with $a \in \mathbb{R}_+$ and $\underline{b} \in \mathbb{R}^m$ originates from the mother wavelet $\psi_{t,\beta}$ by dilation and translation.
The definition of the 'half' Clifford-Jacobi CWTs can be rewritten in frequency space as

$$F^{\pm}_{t,\beta}(a,\underline{b}) = (2\pi)^{m/2}\ a^{m/2}\ \mathcal{F}\left[\left(\mathcal{F}[\psi_{t,\beta}](a\underline{\xi})\right)^{\dagger}\mathcal{F}[f^{\pm}](\underline{\xi})\right](-\underline{b}) \quad . \tag{4.3}$$

We now prove that the 'half' Clifford-Jacobi CWT $T^{+}_{t,\beta}$, respectively $T^{-}_{t,\beta}$, maps the Hardy space $H^2(\mathbb{R}^m)$, respectively $H^2(\mathbb{R}^m)^{\perp}$, isometrically into a weighted L_2-space on $\mathbb{R}_+ \times \mathbb{R}^m$.
To that end we calculate

$$\int_{\mathbb{R}^m}\int_0^{+\infty}\left(F^{\pm}_{t,\beta}(a,\underline{b})\right)^{\dagger}\ G^{\pm}_{t,\beta}(a,\underline{b})\ \frac{da}{a^{m+1}}\ dV(\underline{b}) \quad . \tag{4.4}$$

In view of (4.3) and the Parseval formula, we obtain

$$\int_{\mathbb{R}^m}\int_0^{+\infty}\left(F^{\pm}_{t,\beta}(a,\underline{b})\right)^{\dagger}\ G^{\pm}_{t,\beta}(a,\underline{b})\ \frac{da}{a^{m+1}}\ dV(\underline{b}) \ =$$
$$(2\pi)^m\int_{\mathbb{R}^m}\left(\mathcal{F}[f^{\pm}](\underline{\xi})\right)^{\dagger}\left(\int_0^{+\infty}\mathcal{F}[\psi_{t,\beta}](a\underline{\xi})\ \left(\mathcal{F}[\psi_{t,\beta}](a\underline{\xi})\right)^{\dagger}\ \frac{da}{a}\right)\ \mathcal{F}[g^{\pm}](\underline{\xi})\ dV(\underline{\xi}) \quad .$$

By means of the substitution

$$\underline{\xi} = \frac{r}{a}\ \underline{\eta} \quad , \quad \underline{\eta} \in S^{m-1} \quad ,$$

the integral between brackets becomes

$$\int_0^{+\infty}\mathcal{F}[\psi_{t,\beta}](a\underline{\xi})\ \left(\mathcal{F}[\psi_{t,\beta}](a\underline{\xi})\right)^{\dagger}\ \frac{da}{a} = \int_0^{+\infty}\mathcal{F}[\psi_{t,\beta}](r\underline{\eta})\ \left(\mathcal{F}[\psi_{t,\beta}](r\underline{\eta})\right)^{\dagger}\ \frac{dr}{r} \ .$$

Next, using expression (4.2) we find

$$\int_0^{+\infty}\mathcal{F}[\psi_{t,\beta}](a\underline{\xi})\ \left(\mathcal{F}[\psi_{t,\beta}](a\underline{\xi})\right)^{\dagger}\ \frac{da}{a} = c_1 + c_2\ \underline{\eta}$$

with

$$c_1 = \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2\int_0^{+\infty} r^{-m-2\beta-1}\left(\left(K_{m/2+\beta+t}(r)\right)^2 + \left(K_{m/2+\beta+t+1}(r)\right)^2\right)\ dr$$

and

$$c_2 = -2i\ \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2\int_0^{+\infty} r^{-m-2\beta-1}K_{m/2+\beta+t+1}(r)\ K_{m/2+\beta+t}(r)\ dr \quad .$$

Note that c_1 and c_2 are finite constants, since the modified Bessel functions of the second kind K_ν with $\mathrm{Im}(\nu) = 0$, have the following limiting behaviour:

$$K_\nu(x) \approx \frac{\pi}{2\sin(\pi|\nu|)}\left(\frac{1}{-|\nu|!}\right)\left(\frac{2}{x}\right)^{|\nu|} \quad \text{for } x \to 0$$

and

$$K_\nu(x) \approx \left(\frac{\pi}{2x}\right)^{1/2} \exp(-x) \quad \text{for } x \to \infty \quad ,$$

and since moreover we consider $0 < t < (-2\beta - m)/2$.
Consequently, the integral (4.4) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^m} \int_0^{+\infty} \left(F^\pm_{t,\beta}(a, \underline{b})\right)^\dagger \, G^\pm_{t,\beta}(a, \underline{b}) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \\
&= \quad (2\pi)^m \, c_1 < \mathcal{F}[f^\pm], \mathcal{F}[g^\pm] > \\
&\quad + (2\pi)^m \, c_2 \int_{\mathbb{R}^m} \left(\mathcal{F}[f^\pm](\underline{\xi})\right)^\dagger \, \underline{\eta} \, \mathcal{F}[g^\pm](\underline{\xi}) \, dV(\underline{\xi}) \quad .
\end{aligned}$$

As

$$\begin{aligned}
\underline{\eta} \, \mathcal{F}[g^\pm](\underline{\xi}) &= \underline{\eta} \, P^\pm \, \mathcal{F}[g](\underline{\xi}) \\
&= \mp i \, P^\pm \, \mathcal{F}[g](\underline{\xi}) \\
&= \mp i \, \mathcal{F}[g^\pm](\underline{\xi}) \quad ,
\end{aligned}$$

this integral can be further simplified to

$$\begin{aligned}
&\int_{\mathbb{R}^m} \int_0^{+\infty} \left(F^\pm_{t,\beta}(a, \underline{b})\right)^\dagger \, G^\pm_{t,\beta}(a, \underline{b}) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \\
&= \quad (2\pi)^m \, (c_1 \mp i c_2) < f^\pm, g^\pm > \quad .
\end{aligned}$$

Hence we can define the following inner products on the spaces of transforms

$$[F^\pm_{t,\beta}, G^\pm_{t,\beta}] = \frac{1}{C^\pm} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F^\pm_{t,\beta}(a, \underline{b})\right)^\dagger G^\pm_{t,\beta}(a, \underline{b}) \, \frac{da}{a^{m+1}} \, dV(\underline{b})$$

with

$$C^\pm = (2\pi)^m \, (c_1 \mp i c_2) \quad .$$

These inner products satisfy the Parseval formulae:

$$[F^\pm_{t,\beta}, G^\pm_{t,\beta}] = < f^\pm, g^\pm >$$

which implies that the 'half' Clifford-Jacobi CWTs $T^\pm_{t,\beta}$ are isometries, as it should be.

The twin transforms $T^\pm_{t,\beta}$ may be combined into a 'complete' Clifford-Jacobi CWT acting on $L_2(\mathbb{R}^m, dV(\underline{x}))$. Indeed, for $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ we put

$$\begin{aligned}
T_{t,\beta}[f](a, \underline{b}) &= F_{t,\beta}(a, \underline{b}) \\
&= < \psi^{a,\underline{b}}_{t,\beta} \, , \, f > \\
&= < \psi^{a,\underline{b}}_{t,\beta} \, , \, f^+ > + < \psi^{a,\underline{b}}_{t,\beta} \, , \, f^- > \\
&= F^+_{t,\beta}(a, \underline{b}) + F^-_{t,\beta}(a, \underline{b}) \quad .
\end{aligned}$$

Remark 4.4. The CK-extension technique also works for the more general weight function

$$G(\underline{x}) = (1 + \underline{x})\ (1 + |\underline{x}|^2)^\beta\ P_k(\underline{x}) \quad , \quad \beta \in \mathbb{R}$$

with $P_k(\underline{x})$ an arbitrary but fixed left inner spherical monogenic of order k. In this way, one obtains the so-called generalized special Clifford-Jacobi polynomials $J_{t,\beta,k}(\underline{x})$ ($t = 0, 1, 2, \ldots$) which are the appropriate building blocks for the generalized Clifford-Jacobi wavelets given by

$$\begin{aligned} \psi_{t,\beta,k}(\underline{x}) &= (1 + |\underline{x}|^2)^\beta\ (1 + \underline{x})\ J_{t,\beta+t,k}(\underline{x})\ P_k(\underline{x}) \\ &= (-1)^t\ \partial_{\underline{x}}^t \Big((1 + |\underline{x}|^2)^{\beta+t}\ (1 + \underline{x})\ P_k(\underline{x}) \Big) . \end{aligned}$$

Naturally, these wavelets are not invariant under the rotation group $Spin(m)$. Hence this group must be taken into consideration when defining the associated CWT.

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